A NEW REGULARIZATION OF THE
PLANAR PROBLEM OF THREE BODIES*

JOERG WALDVOGEL**

University of Texas at Austin, Austin, Tex., U.S.A.

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Abstract. A new method of simultaneously regularizing the three types of binary collisions in the planar problem of three bodies is developed: The coordinates are transformed by means of certain fourth degree polynomials, and a new independent variable is introduced, too. The proposed transformation is in each binary collision locally equivalent to Levi-Civita’s transformation, whereas the singularity corresponding to a triple collision is mapped into infinity. The transformed Hamiltonian is a polynomial of degree 12 in the regularized variables.

1. Introduction

The differential equations of the problem of three bodies, written in rectangular coordinates, are an adequate description of the motion of the three point masses when none of the bodies collide. If a collision takes place, at least one of the three mutual distances vanishes, and the equations of motion are no longer valid; they are singular.

In the case of a collision between two bodies (binary collision) one of the distances vanishes. The corresponding singularities in the solutions of the equations of motion are branch points of order 3, as was shown by Levi-Civita. These singularities can be removed from the differential equations and their solutions by classical regularizing transformations. The simplest among these was found by Levi-Civita (1906); it regularizes the collision between two bodies at each eventual recurrence. Such transformations are referred to as local regularizations. As opposed to this, the simultaneous regularization of all possible binary collisions in a gravitational problem will be called ‘global’ regularization, in agreement with G. D. Birkhoff’s terminology for the restricted problem of three bodies.

On the other hand, it is not possible to regularize a collision of more than two point masses since the corresponding solutions generally have branch points of infinite order. Therefore, the triple collision (Siegel, 1941) is excluded from most consideration in this paper.

In global regularizations the transformed equations of motion are often particularly simple: Levi-Civita’s transformation applied to the problem of two bodies regularizes the only possible collision; the transformed differential equations are linear with constant coefficients, and their solutions are holomorphic in the whole complex plane.

For the restricted problem of three bodies global regularization is achieved by

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** Assistant Professor, Department of Aerospace Engineering and Engineering Mechanics.

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Birkhoff’s transformation or by Thiele’s transformation. Another global regularization of this problem is named after Lemaître; in a modified form it has been used by Arenstorf (1963) to derive interesting global properties of the solutions of the restricted problem of three bodies.

Lemaître (1954) found a global regularization of the general three-body problem based on the classical reduction of this problem (elimination of the nodes). The transformations and final equations, however, are complicated and in most cases not well suited for numerical purposes. For further references see Szebehely (1967).

In this paper a new, simple global regularization of the planar problem of three bodies is developed, based on Levi-Civita’s transformation. The right-hand sides of the regularized equations of motion are polynomials in every dependent variable. These variables show advantages when the planar problem of three bodies is numerically integrated.

First Levi-Civita’s transformation is reviewed using the Hamiltonian formalism and complex notation for vectors. An extension of this transformation is then applied to the canonical equations of the planar problem of three bodies.

2. Levi-Civita’s Regularization

We consider a particle of infinitesimal mass moving in a plane. Its position is described by the complex vector \( x = x_1 + i x_2 \), where \( x_1, x_2 \) are the cartesian coordinates of the particle with respect to a coordinate system centered at 0. The conjugate momenta are \( p_1, p_2 \), and the ‘complex momentum’ is \( p = p_1 + i p_2 \).

The particle is subject to the Newtonian attraction of a central body located at 0. Other conservative forces whose potentials have no singularities at 0 may be present. The Hamiltonian \( H(x, p) \) of the system may be written as a (non-analytic) function of the complex variables \( x \) and \( p \). The equations of motion in canonical form are

\[
\frac{dx_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial x_j}, \quad j = 1, 2. \tag{1}
\]

The first step of Levi-Civita’s regularization consists of introducing new coordinates \( \xi_1, \xi_2 \) by the conformal transformation

\[
x = \xi^2, \tag{2}
\]

where \( \xi = \xi_1 + i \xi_2 \). In order to preserve the canonical form (1) of the equations of motion, new momenta \( \pi_1, \pi_2 \) and the complex quantity \( \pi = \pi_1 + i \pi_2 \) are introduced such that the transformation to the new variables is canonical.

For defining the new momenta in more general cases we consider the conformal transformation

\[ x = x(\xi) \text{ or } \xi = \xi(x), \]

where \( x(\xi) \) is an analytic function. Introducing the generating function \( G(x_1, x_2, \pi_1, \pi_2) \) we have

\[
\xi_j = \frac{\partial G}{\partial \pi_j}, \quad p_j = \frac{\partial G}{\partial x_j}, \quad j = 1, 2. \]