Abstract. A second-order libration solution of the Ideal Resonance Problem is constructed using a Lie-series perturbation technique. The Ideal Resonance Problem is characterized by the equations

\[- F = B(x) + 2\mu^2 A(x) \sin^2 y, \]
\[\dot{x} = - F_y, \quad \dot{y} = F_x, \]

together with the property that \(B_x\) vanishes for some value of \(x\).* Explicit expressions for \(x\) and \(y\) are given in terms of the mean elements; and it is shown how the initial-value problem is solved. The solution is primarily intended for the libration region, but it is shown how, by means of a substitution device, the solution can be extended to the deep circulation regime. The method does not, however, admit a solution very close to the separatrix. Formulae for the mean value of \(x\) and the period of libration are furnished.

1. Introduction

It is well known that the presence of resonance in a dynamical system often precludes the construction of an analytic solution within the domain of resonance; while away from the resonance region a solution may quite simply be found. The mathematical difficulty encountered in problems of resonance is the occurrence of small divisors in the solutions formulated from the standard classical theories. The usual device for avoiding the troublesome small divisors is that suggested by Bohlin (1889), and further developed by Poincaré (1893). Rather than seeking a solution in powers of some small parameter associated with the problem, as in the classical approach, a solution is sought in powers of the square root of that small parameter.

As a starting point for the study of resonance problems, Garfinkel (1966) formulated and gave the first formal discussion of the Ideal Resonance Problem. The problem has the Hamiltonian form

\[- F = B(x) + 2\mu^2 A(x) \sin^2 y, \]
\[\dot{x} = - F_y, \quad \dot{y} = F_x. \tag{1} \]

The canonically conjugate momentum and co-ordinate variables are, respectively, \(x, y\); and \(\mu^2\) is a small positive constant parameter. (In the author’s previous publications \(\mu\) has been used in place of \(\mu^2\); the new choice avoids the use of fractional powers.) The variables are chosen so that \(A > 0\) for all values of \(x\) (c.f. Garfinkel,

* In this complete second-order theory, the equation for the momentum \(x\) is, in fact, correct to third order in the small parameter \(\mu\). This state of affairs is somewhat different from the usual situation.
The resonance phenomenon is associated with the vanishing of \( B^{(1)} = B_x \) for some value(s) of \( x \).

System (1) may be thought of as a kind of perturbed simple pendulum. Indeed, the phase-plane configuration of the Ideal Resonance Problem is analogous to that of the simple pendulum; the separatrix dividing the plane into regions of libration and circulation in a like fashion.

It is a one-degree-of-freedom system and is therefore soluble by quadratures; but this is not necessarily a straightforward nor a useful approach. The Ideal Resonance Problem may be used as a first approximation to a number of more complicated practical resonance problems; it is desirable, therefore, to construct an analytic solution. An analogy may be made with the two-body problem, which is used as a first approximation to many more complex problems – artificial satellite theory and planetary theory, for example.

In his original publication, Garfinkel employs the von Zeipel perturbation technique to obtain a first-order asymptotic solution; i.e. to \( O(\mu) \). A second paper, by Garfinkel et al. (1971), hereafter referred to as Paper 3, considers the higher order terms of the solution; and recursive relations are constructed, from which, given the time and the values of certain constants of the motion, the solution to any desired order can be computed. The solution is general and valid throughout the phase plane.

An alternative method of solution has been given, (Jupp, 1969, Paper 1), based upon a choice of variables proposed by Poincaré (1893). In Paper 1 the author suitably develops and adapts Poincaré’s variable for the Ideal Resonance Problem, and the solution is given in an implicit form to \( O(\mu) \). There the solution is restricted to the libration regime. In a second paper, Jupp (1970), [Paper 2–], the general nature of the higher order terms is discussed. It is indicated that, to any order, the solution is free from singularities; further it is demonstrated that only a small number of different elliptic functions can be expected to appear in the solution.

The present communication seeks to complete this latter study of the Ideal Resonance Problem. Formulae for \( x \) and \( y \) are here given explicitly to \( O(\mu^3) \) and \( O(\mu^2) \), respectively, for the libration region. It is shown how by means of a simple procedure, the solution may be extended into the deep circulation region. The complete first-order circulation solution is given explicitly. A number of general results are stated, in parallel with results obtained in Papers 1 and 2.

While the method described here is not applicable throughout the entire phase plane (the vicinity of the separatrix and the region of classical circulation must be excluded), a property enjoyed by the global solution of Paper 3, it has a number of important advantages, enumerated below.

1. There is no loss of an order of magnitude on differentiation of the determining function \( S \) and the new Hamiltonian \( F' \) with respect to the momentum. Consequently, it is not necessary to calculate \( S_{n+1} \) in order to construct the \( n \)th order solution.

2. Solving the initial-value problem is perfectly straightforward (Section 7). In contrast, the inversion of Equation (112) of Paper 3 leads to singularities in the libration regime.