ON THE NUMBER OF EFFECTIVE INTEGRALS IN GALACTIC MODELS

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Abstract. Three different numerical techniques are tested to
determine the number of integrals of motion in dynamical systems with
three degrees of freedom.

(1) The computation of the whole set of Lyapunov Characteristic
Exponents (LCE).

(2) The triple sections in the configurations space.

(3) The Stine-Noid box-counting technique.

These methods are applied to a triple oscillator with coupling terms
of the third order. Cases are found for which one effective integral
besides the Hamiltonian subsists during a very long time. Such orbits
display simultaneously chaotic and quasi-periodic motion, according to
which coordinates are considered.

As an application, the LCE procedure is applied to a triaxial
elliptical galaxy model. Contrary to similar 2-dimensional systems,
this 3-dimensional one presents noticeable zones in the phase space
without any non-classical integral.

1. Introduction

The last two decades have seen the publication of a great deal of works
devoted to the problems of stochastic orbits and of the existence, or
absence, of the so-called non-classical integrals, not only in galactic
models, but also in the general frame of non-linear differential
equation systems.

The integration of the equations of motion in either axisymmetric
or triaxial galactic models shows that orbits are often characterized
by one or two isolating integrals apart from the Hamiltonian. This
characteristic is used in the construction of self-gravitating models
(Schwarzschild, 1979; Richstone, 1980). The considered systems being
generally analytically non-integrable, it is advisable, following
Schwarzschild, to employ for these non-classical integrals the term of
EFFECTIVE integrals. This term applies to any function in phase space
being substantially constant along an orbit for a Hubble time span, and
isolating.

The experience acquired in the study of models with two degrees
of freedom (2DF) shows that in most cases zones without non-classical integrals in phase space coexist with integrable ones. Although much less numerous, the recently executed computations in 3DF systems indicate a similar situation.

It is therefore as important to study the non-integrable (or chaotic) zones in 3DF galactic models as it was in axisymmetric ones. Moreover, an important feature of 3DF dynamical systems is the fact that they can, theoretically at least, reveal three different situations from the effective integrals point of view: 0, 1, or 2 integrals may be present apart from the Hamiltonian. The intermediate case, where one non-classical integral coexist with the Hamiltonian, deserves particular attention: if one effective non-classical integral can hold for a long time span, it means that a trajectory can in some cases present a chaotic aspect, while being in other respects submitted to some constraints as to the accessible region in phase space.

C. Froeschlé, who broke new ground in a whole set of techniques for the study of 3DF problems, turned his attention to this question a long time ago (Froeschlé, 1970a, b, c), as much in the three-dimensional three-body restricted problem as in an algebraic four-dimensional mapping. He observed a quasi-simultaneous disappearance of both non-classical integrals: either the system has two non-classical integrals, or it has none. This behaviour is probable: as soon as two coordinates undergo a chaotic evolution, the two others (in a four-dimensional space of section) are 'contaminated' by the coupling term, however weak it is, and sooner or later also behave in a chaotic way (see also Hénon, 1983). On the other hand, a different thesis was supported by Contopoulos et al. (1978), who also found cases with only one non-classical integral, using the techniques of the maximal Lyapunov Characteristic Exponent (however, as it will be seen below, this technique is not efficient enough) and of formal integrals. Such results were confirmed and ascertained by Benettin et al. (1980, Section 8) by computing all the LCE.

For our present concern, it is important to see whether the contamination referred to above is rapid or not comparatively to the age of the Universe, in cases of models which simulate more or less realistically galactic potentials. This question was grappled with by Martinet and Magnenat (1981; hereafter MM) in the study of orbital behaviour in the Hamiltonian

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + A\dot{x}^2 + B\dot{y}^2 + C\dot{z}^2) - \varepsilon x z - \eta y z$$

(1)

Where $A = 0.9$, $B = 0.4$, $C = 0.225$.

Making use of the surface of section technique and stereoscopic projections, MM observed the progressive deformation and dissolution of an invariant torus when one perturbation parameter is increased, all other things being kept constant. Particularly, the case $\varepsilon = 0.5$, $h = 0.00765$ and initial conditions

$$\tilde{x}_0 = -0.0378, \tilde{y}_0 = 0.025, \tilde{z}_0 = 0, \tilde{x}_0 = \tilde{y}_0 = 0$$