Notes

Saint-Venant’s Problem for Anisotropic Circular Cylinders

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1. Introduction

In [1] a method of solving Saint-Venant’s problem for inhomogeneous and anisotropic elastic beams is presented. This method points out the importance of the auxiliary generalized plane strain problems in the treatment of Saint-Venant’s problem.

Special cases of Saint-Venant’s problem for anisotropic elastic cylinders are considered in [2].

In this paper we consider the Saint-Venant’s problem in the case of anisotropic and homogeneous circular cylinders. In the first part of the paper solutions of the auxiliary generalized plane strain problems are established. Then, extension, bending and torsion problems are solved. The solution is a polynomial of second degree in the Cartesian coordinate $x_i$. In the last part of the paper, we consider the case when the loading acting on one of the ends is statically equivalent to a force $R(R_i)$ and a moment $M(M_i)$. The solution of this problem is a polynomial of degree three in $x_i$.

2. Statement of the Problem

Throughout this paper a rectangular coordinate system $Ox_k$ ($k = 1, 2, 3$) is used. We consider a circular cylindrical beam of homogeneous and anisotropic elastic material which occupies the region $R$ of space, whose boundary is $S$. We suppose that the circular cylinder is bounded by plane ends perpendicular to the generators. The boundary of the generic cross-section $\Sigma$, which is a circle of radius $a$, is denoted by $L$. Throughout this paper the axis $Ox_3$ of our coordinate system will be directed parallel to the generators of the cylinder. The cylinder is assumed to be of length $h$, and one of its bases is taken to lie in the $x_1Ox_2$-plane, while the other is in the plane $x_3 = h$.

Unless otherwise specified, we shall employ the usual conventions: Greek subscripts are understood to range over the integers $(1, 2)$ whereas Latin subscripts are confined to the range $(1, 2, 3)$; summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.
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Let \( u_i \) denote the components of the displacement vector field. Then the components of the infinitesimal strain field are given by

\[
2\varepsilon_{ij} = u_{i,j} + u_{j,i}. 
\]

The stress-strain relations in the case of an anisotropic elastic medium are

\[
t_{ij} = C_{ijrs}e_{rs},
\]

where \( t_{ij} \) are components of the stress tensor and \( C_{ijrs} \) are the components of the elasticity tensor which obey the symmetry relations

\[
C_{ijrs} = C_{jirs} = C_{rei}. 
\]

The equations of equilibrium, in absence of body forces, are

\[
t_{ji,i} = 0, \text{ in } R, 
\]

and the equations of compatibility are

\[
\varepsilon_{ij,rs} + \varepsilon_{rs,ij} - \varepsilon_{ir,j} - \varepsilon_{js,ir} = 0, \text{ in } R. 
\]

The circular cylinder is supposed to be free from lateral loading, so that the conditions on the lateral surface are

\[
t_{a,n} = 0,
\]

where \((n_1, n_2, 0)\) are the direction cosines of the outward normal to the lateral surface.

We assume that the load of the beam is distributed over its ends in a way which fulfills the equilibrium conditions of a rigid body. Let the loading applied on the end located at \( x_3 = 0 \) be statically equivalent to a force \( R(R_i) \) and a moment \( M(M_i) \) such that, for \( x_3 = 0 \), we have the conditions

\[
\int_{S} t_{3a} d\sigma = -R_a, 
\]

\[
\int_{S} t_{33} d\sigma = -R_3, \quad \int_{S} x_a t_{33} d\sigma = e_{3\alpha\beta}M_\beta, \quad \int_{S} e_{3\alpha\beta}x_a t_{33} d\sigma = -M_3, 
\]

where \( \varepsilon_{ij3} \) is the alternating symbol.

The relations between \( t_{ij} \) and \( \varepsilon_{ij} \) must be reversible, hence we can write

\[
\varepsilon_{ij} = S_{ijrs}t_{rs},
\]

where

\[
C_{ijmn}S_{mnrs} = \frac{1}{2} (\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}).
\]

We will have occasion to use four special problems \( P^{(m)} \) \((m = 1, 2, 3, 4)\) of generalized plane strain [1]. In what follows we denote by \( u_i^{(m)}, t_i^{(m)} \) the components of the displacement vector and the components of the stress tensor from the