POWER SERIES SOLUTIONS TO BASIC STATIONARY BOUNDARY VALUE PROBLEMS OF ELASTICITY

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Dedicated to Prof. I. Gohberg on the occasion of his 70th birthday

The four basic stationary boundary value problems of elasticity for the Lamé equation in a bounded domain of $\mathbb{R}^3$ are under consideration. Their solutions are represented in the form of a power series with non-positive degrees of the parameter $\omega = 1/(1 - 2\sigma)$, depending on the Poisson ratio $\sigma$. The "coefficients" of the series are solutions of the stationary linearized non-homogeneous Stokes boundary value problems. It is proved that the series converges for any values of $\omega$ outside of the minimal interval with the center at the origin and of radius $r \geq 1$, which contains all of the Cosserat eigenvalues.

1 Introduction

Let $\Omega$ be a bounded connected open set in $\mathbb{R}^3$ with an infinitely smooth boundary $\Gamma = \partial \Omega$, and $n = (n_1, n_2, n_3)$ be the unit interior normal vector field defined near $\Gamma$. Denote $f|_r = \gamma_0 f$ and $\gamma_0 \partial_k^j f|_r = \gamma_k f$, where $\partial_0 f = \sum_{j=1}^3 n_j \partial_j f$, $\partial_j f = \partial f/\partial x_j$. Let $u$ be the displacement vector $u = (u_1, u_2, u_3)$. We consider the stationary linearized basic boundary value problems of elasticity [7, Ch. III, Sect. 2.2], [16], [19, Sect. 3]:

(i) $\Delta u + \omega \text{grad} \text{div} u = f$ for $x \in \Omega,$

(ii) $T^\omega_k \left( \begin{array}{c} u \\ \text{div} u \end{array} \right) = g$ for $x \in \Gamma,$

Here $\Delta$ is the Laplace operator, $u, f, g$ are vector-functions range in $\mathbb{R}^3$, $\omega = 1/(1 - 2\sigma)$ and $\sigma$ is the Poisson ratio ($-1 < \sigma < 1/2$); $T^\omega_k$ ($k = 1, 2, 3, 4$) are boundary operators describing respectively the four basic boundary value problems of elasticity: with given displacements $u$ ($k = 1$) and stresses ($k = 2$) on the boundary or the problem of hard ($k = 3$) and soft contact ($k = 4$). In the problem of hard contact ($k = 3$) the normal component of the displacement and the tangential component of the stress vector are given on the boundary.
In the problem of soft contact ($k = 4$) the normal component of the stress vector and the tangential component of the displacement are given on the boundary. More precisely,

$$T_1^\omega \left( \begin{array}{c} u \\ \text{div} u \end{array} \right) := \gamma_0 u,$$

$$T_2^\omega \left( \begin{array}{c} u \\ \text{div} u \end{array} \right) := \chi_1 u + (\omega - 1) \gamma_0 (\text{div} u) n,$$

$$T_3^\omega \left( \begin{array}{c} u \\ \text{div} u \end{array} \right) := (\chi_1 u)_\tau + \gamma_0 u_n,$$

$$T_4^\omega \left( \begin{array}{c} u \\ \text{div} u \end{array} \right) := (\gamma_0 u)_\tau - (\chi_1 u)_n - (\omega - 1) \gamma_0 (\text{div} u) n,$$

where $u_n := \sum_{j=1}^3 u_j n_j$ and $u_\tau$ is the tangential component of $u$, i.e. $u_\tau := u - u_n$. $\chi_1$ is a special first order boundary operator defined via the strain vector $X_{1u}$ with components

$$(\chi_1 u)_i = \gamma_0 \left( \sum_{j=1}^3 (\partial_i u_j + \partial_j u_i) n_j \right) \quad (i = 1, 2, 3).$$

We use the unified notation $T_k^\omega$, although the operators $T_1^\omega$ and $T_3^\omega$ do not depend on $\omega$.

We show that solutions of the four boundary value problems of elasticity (1) may be represented in the form of the power series in $\varepsilon := \omega^{-1}$ (see (7) below) strongly convergent in an appropriate Sobolev space for any values of $\omega$ outside of the minimal interval with the center at the origin and of radius $r \geq 1$, which contains all of the Cosserat eigenvalues. Here $r$ depends on the domain $\Omega$ and the number of the boundary problem $k$. In particular, the values of $r$ for some domains were found in the classical papers by Eugène and François Cosserat [2]-[5] (see also [19]). For instance, for the ball $\Omega := \{ x \in \mathbb{R}^3 : |x| < R \}$ we have $r = 3, 1$ in the cases of the first, second ($k = 1, 2$) problem (1), respectively.

The “coefficients” of the series are solutions of the stationary linearized completely non-homogeneous Stokes problem. According to [24, Ch. I, Sect. 6], since many efficient algorithms are known for solving the Stokes problem, and since the discretization of (1) leads to a very ill-conditioned matrix for large $\omega$ (materials with the Poisson ratio $\sigma$ near to 1/2), one can try to compute $u$ for large $\omega$ by using Stokes equations. The advantage of the Stokes problem is that the corresponding hydrodynamic potentials lead to regular Fredholm integral equations on the boundary while the elastic potentials generate singular Fredholm equations.

The obtained power series gives us a theoretical foundation for the well-known “optical-polarization” method or the “photo-elasticity” method of determining the static stress of a three-dimensional body. As is known, this method allows one with the help of experiments to find stresses for “models”, i.e. bodies prepared from special plastics. These stresses are considered as approximations of stresses for the bodies of natural materials. The Poisson ratios for materials used nowadays as model materials are approximately 1/2 ($\omega \approx \infty$). Using the first term of the power series, we can calculate (instead of finding experimentally) static stresses for such “models”, as well as for bodies of natural materials.