The maximal number of non-isomorphic abelian groups of order $n$

By

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Denote by $a(n)$ the number of non-isomorphic abelian groups of order $n$. As is well known ([9], page 51), $a(n)$ is a multiplicative function, and $a(p^2) = P(p)$, the number of unrestricted partitions of $p$.

On average, $a(n)$ behaves like a constant, on account of the asymptotic formula ¹)

$$\sum_{n \leq x} a(n) = \prod_{p=2}^{\infty} \zeta(p) \cdot x + O(x^{1/2})$$

(Erdős and Szekeres [2]). For individual $n$'s, $a(n)$ was estimated in 1947 by Kendall and Rankin [3]; their result was sharpened in 1970 by E. Krätzel [4], who proved ²)

$$\limsup_{n \to \infty} \log a(n) \cdot \left( \frac{\log n}{\log \log n} \right)^{-1} = \frac{1}{3} \log 5.$$ 

We denote, as usual, the number of primes $p \leq x$ by $\pi(x) = \sum \log p$.

In this note we sharpen Krätzel's result in proving the following

**Theorem.** Let $A = A(n)$ be the smallest integer such that

$$\theta(A) \geq \frac{1}{3} \log n.$$ 

Then for $n \to \infty$ the estimate

$$\log a(n) \leq \log 5 \cdot \pi(A) + O((\log n)^\theta),$$

holds with

$$\theta = \frac{\log 121}{\log 125} < 0.994,$$

and there are infinitely many integers $n$ for which one has

$$\log a(n) = \log 5 \cdot \pi(A).$$

¹) Several authors improved this formula; the best results are due to P. G. Schmidt [8]; further references are to be found in P. G. Schmidt's paper.

²) In A. G. Postnikov's book [5] a general theorem [by A. A. Drozdovoi and G. A. Freiman] is proved, which implies $\log a(n) \leq C \cdot \frac{\log n}{\log \log n} \cdot \left( 1 + O\left( \frac{\log \log n}{\log \log n} \right) \right)$; by using Krätzel's results, one derives $C = \frac{1}{3} \log 5$. 
If we use the prime number theorem in the form
\[ \pi(x) = \text{li} x + R_1(x), \quad \vartheta(x) = x + R_2(x) \]
with remainder terms
\[ R_i(x) \ll x \cdot \exp\left( -c_1 \cdot \sqrt{\log x} \right), \]
then
\[ A(n) = \frac{1}{8} \log n + O(\log n \cdot \exp\left( -c_1 \cdot \sqrt{\log \log n} \right)), \]
and some easy calculations immediately give the

Corollary 3). If \( n \to \infty \), then
\[ (2') \log a(n) \leq \log 5 \cdot \text{li} \left( \frac{1}{4} \log n \right) + O(\log n \cdot \exp\left( -c_2 \sqrt{\log \log n} \right)). \]
There are infinitely many values of \( n \) such that
\[ (3') \log a(n) \geq \log 5 \cdot \text{li} \left( \frac{1}{4} \log n \right) + O(\log n \cdot \exp\left( -c_2 \sqrt{\log \log n} \right)). \]
Formula (3) is trivially true for the sequence \( n_r = (2 \cdot 3 \cdot 5 \ldots p_r)^4 \)
with
\[ a(n_r) = P(4)^r = 5^r, \quad A(n_r) = p_r, \quad \pi(A) = r. \]

The proof of (2) uses a result of Erdős [1],
\[ (4) \log P(\alpha) \leq c_3 \cdot \sqrt{\alpha}, \]
where \( c_3 = \pi \cdot \sqrt{\frac{3}{2}} \). Because of this estimate the function \( q(\alpha) = \alpha^{-1} \log P(\alpha) \) takes a maximal value somewhere. In fact it is sufficient to compute \( q(\alpha) \) for \( \alpha \leq 42 \) in order to determine the largest and second largest value of \( q(\alpha) \); these are \( q(4) = \frac{1}{4} \log 5 \) and \( q(6) = \frac{1}{4} \log 11 \) respectively. Therefore we have
\[ (5) \log P(\alpha) \leq \frac{1}{8} \log 5 \cdot \alpha \quad \text{for all } \alpha, \]
\[ (6) \log P(\alpha) \leq \frac{1}{8} \log 5 \cdot \alpha = \frac{1}{4} \log 11 \cdot \alpha \quad \text{for } \alpha \neq 4, \]
with the constant \( \theta \) specified in the theorem. The estimate (5) was already used by Krätzel [4].

Let \( n = \prod_{p} p^{a(p)} \) and
\[ (7) \delta = \frac{1}{4} \cdot \frac{\log 5}{\log A} \]
with \( A = A(n) \) as defined in the theorem.

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3) The corresponding problem for the function \( \tau(n) \), the number of divisors of \( n \), was solved by S. Ramanujan in his paper on "Highly composite numbers". Proc. London Math. Soc. (2) 14, 347–409 (1915).

4) It is only for this bound that we need the value of \( c_3 \). It is possible to reduce the bound 42 to 12 by using Krätzel's estimate ([4], p. 274)
\[ \log P(\alpha) < \frac{1}{4} \log 5 + \log \left( \frac{8\pi^2}{3 \log^2 5} \right) - \log \alpha, \quad \alpha \geq 11, \]
for \( 13 \leq \alpha \leq 42 \).