STABILITY OF OUTER PLANETARY SYSTEMS*

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Abstract. The conditions for stability in the Liapunov–Hill sense of outer planetary systems are given in terms of radii of planetary orbits. The outer planets of the solar system are found stable and the possible existence of other than the presently known planets between Jupiter and Pluto are indicated. The existence of other planetary systems with arbitrary mass ratios of the primaries is suggested, and the stability conditions for such systems are derived.

1. Introduction

The model of the restricted problem of three bodies is accepted to establish stability regions for planetary orbits outside the primaries. For the conventions, notations and arrangement of the primaries and of the collinear equilibrium points see for instance Szebehely (1967). The mass-parameter is \( \mu = m_2/(m_1 + m_2) \) where \( m_1 \approx m_2 \) are the masses of the primaries. The locations of the primaries (\( P_1 \) and \( P_2 \)) and of the equilibrium points (\( L_1, L_2 \) and \( L_3 \)) on the \( x \)-axis are given here for the solar system (Sun and Jupiter as primaries, \( \mu = 0.000 \, 953 \, 875 \, 4 \)) and for the Copenhagen problem (\( \mu = \frac{1}{2} \)) for general orientation. For the solar system \( P_1(-0.999 \, 046), \, P_2(0.000 \, 954), \, L_1(-1.068 \, 831), \, L_2(-0.932 \, 366) \) and \( L_3(1.000 \, 397) \). For the Copenhagen problem \( P_1(-0.5), \, P_2(0.5), \, L_1(-1.198 \, 406), \, L_2(0) \) and \( L_3(1.198 \, 406) \).

The Jacobian constant is the characteristic parameter of the dynamical system and it is defined by \( C = 2\Omega - v^2 \), where \( \Omega \) is the modified potential function and \( v \) is the velocity relative to the synodic system, rotating with the primaries. The potential function is given by

\[
\Omega = (\frac{1}{2})[(1 - \mu) r_1^2 + \mu r_2^2] + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2},
\]

where \( r_1 \) and \( r_2 \) are the distances between the primaries and the outer planet.

The Jacobian constant corresponding to the equilibrium point, \( L_1 \), is obtained by substituting in its defining equation \( v = 0 \) and \( \Omega(L_1) \). In this way the well known and tabulated function \( C_1 = C_1(\mu) \) may be obtained.

Since the orbit of the outer planet cannot cross its own zero velocity curve, it cannot become a satellite of \( m_1 \) or \( m_2 \) as long as its Jacobian constant \( C \) is larger than \( C_1 \). Consequently, the condition for Hill’s (1878) type stability is that the measure of


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stability, defined by
\[ S = \frac{C - C_1}{C_1} \]  \hspace{1cm} (2)

should have a positive value.

For instance, Uranus' Jacobian constant is \( C = 4.104 \) and the critical value of the Jacobian constant for the Sun–Jupiter system is \( C_1 = 3.038 \). Consequently, \( S = 0.351 > 0 \), demonstrating the Hill's type stability of Uranus. The measures of stability are given for the planets and natural satellites of the solar system by Szébehely (1978).

2. Condition for Stability

For the general case of arbitrary value of the mass parameter, \( C_1(\mu) \) and \( C \) are determined in this section and the inequality \( C_1 < C \) is used to establish the condition for stability. Members of Strömgren's (1924) class I periodic orbits describe outer planetary orbits. These orbits are generated from circular orbits of large radii \( (r \gg 1) \) having the synodic velocity
\[ v = \left( \frac{1}{r} \right)^{1/2} - r. \]  \hspace{1cm} (3)

The orbits are direct in the sidereal system (i.e., co-rotational with the primaries) and retrograde in the synodic system. The radius of the generating circular orbit, \( r \), is measured from the origin of the coordinate system. The relations between \( r \) and the previously mentioned distances \( r_1 \) and \( r_2 \) are \( r_1 = r - \mu \) and \( r_2 = r - \mu + 1 \). Substitution in Equation (1) allows the computation of the Jacobian constant of an outer planet on near-circular orbit:
\[ C = \frac{2\mu}{r + 1 - \mu} + \frac{2(1 - \mu)}{r - \mu} - \frac{1}{r} + 2\sqrt{r}. \]  \hspace{1cm} (4)

The critical value of the Jacobian constant, \( C_1 \) depends only on the mass parameter as mentioned before. This relation may be approximated for small values of \( \mu \) by
\[ C_1 = 3 + 9(\mu/3)^{2/3} - 11(\mu/3) \]  \hspace{1cm} (5)

This approximation gives an accuracy of four decimals for \( \mu \approx 0.01 \) and considering only the first two terms on the right-hand side of Equation (5) offers two correct decimals for \( \mu \approx 0.001 \). For better accuracy and at larger values of \( \mu \), tabulated values are to be used.

The condition for stability now becomes
\[ C_1(\mu) \leq C(\mu, r), \]  \hspace{1cm} (6)

from which the critical value of \( r > 1 \) may be computed for a given value of \( \mu \).

The Jacobian constant for an outer planet as given by Equation (4) may also be approximated for small values of \( \mu \). Neglecting the first and second terms in Equation