TWO-BODY MOTION UNDER THE INVERSE SQUARE CENTRAL FORCE AND EQUIVALENT GEODESIC FLOWS

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Abstract. In this paper the fixed energy surfaces for the two-body problem for parabolic and, in particular, hyperbolic motion are completely determined by utilizing an earlier work of J. Moser. The characterization of these fixed energy manifolds yields the explicit solutions to the above problems in an elementary way for arbitrary dimensions.

1. Introduction

It is known that the Kepler problem on a fixed energy surface is equivalent to a linear system of differential equations. For example, one may achieve this linearization in two dimensions by Levi-Civita’s regularization procedure or in three dimensions by using quaternions – due to Kustaanheimo and Stiefel (Stiefel and Scheifele, 1971). For the case of negative energy $H$ (i.e., elliptic motion) the linearization can also be accomplished, for arbitrary dimensions, by constructing a canonical map of the fixed energy surface into the unit tangent bundle of a sphere – due to J. Moser (1970). In this manner there is established an equivalence of the Kepler motion with the geodesic flow on the unit sphere. The solutions to this problem can then be obtained in any dimension.

In this note we consider the cases of Kepler motion for $H>0$ (i.e., hyperbolic motion) and $H=0$ (i.e., parabolic motion). In particular, for $H>0$ we shall establish the equivalence of this motion with the geodesic flow on a two sheeted hyperboloid embedded in a Lorentz space (see Subsection 2a). Furthermore, an explicit mapping takes one sheet of the hyperboloid isometrically into the Lobachevsky space. The problem can then be reduced to a study of the geodesic flow on the Lobachevsky plane of Gaussian curvature $K=-1$ just as $H<0$ leads to a study of the geodesic flow on the unit sphere of curvature $K=+1$. For $H=0$ we will establish an equivalence of this motion with the linear geodesic flow in Euclidian space (see Subsection 2c). Similar results were indicated by Osipov (1972), but we feel that our use of the hyperboloid yields a more elementary and direct treatment.

There are two distinct advantages of our approach over the Levi-Civita and quaternion methods. The first is a direct generalization to arbitrary dimensions and the second is the relative ease with which the solutions can be read off. However, for the simplicity of exposition we will present our results for the two-dimensional cases only. This method can also be directly applied to the case of planar motion in an...
inverse square central repelling field (see Subsection 2c). This motion is, in fact, closely related to Kepler motion for \( H > 0 \).

It is to be mentioned that the hodograph representations, as conceived by Hamilton (1845) and Möbius (1843), can be used to motivate the construction of the equivalent geodesic flows and therefore will be considered in Subsections 2a, c.

2. Results

2A. Kepler motion for \( H > 0 \)

Consideration of a relative coordinate system with the central mass at the origin allows the equations of motion and the corresponding Hamiltonian to be written in the form

\[
\frac{d^2q}{dt^2} = -\frac{q}{|q|^3}, \quad (1)
\]

\[
H = \frac{1}{2}|p|^2 - |q|^{-1}, \quad (2)
\]

where the 2-vectors \( p, q \) represent the position and velocity respectively of the second mass point. If we let \( c \) be a positive constant, then the energy surface \( H = c \) turns out to be topologically equivalent to the tangent bundle of one sheet of the hyperboloid \( \mathcal{H}_\pm : -\xi_0^2 + \xi_1^2 + \xi_2^2 = -1 \) (\( +, - \) refer to the upper and lower sheets respectively), embedded in a Lorentz space \( \mathbb{L}^3 \) (defined below) after the exclusion of one of the points \( \xi^+ \cdot - = (+1, 0, 0) \). The geodesics of \( \mathcal{H}_\pm \) passing through \( \xi^+ \cdot - \) correspond to collision orbits. More precisely, we will prove the following:

**THEOREM 1.** The energy surface \( H = c \) can be mapped topologically one to one into the tangent bundle of \( \mathcal{H}_+ \) or \( \mathcal{H}_- \). This mapping is onto the tangent bundle of \( \mathcal{H}_+ \) or \( \mathcal{H}_- \) (each punctured at the respective points \( \xi^+ \) or \( \xi^- \)). Furthermore, the flow defined by the Kepler problem is mapped into the geodesic flow on \( \mathcal{H}_+ \) or \( \mathcal{H}_- \) after a change of the independent variable.

The Kepler flow is regularized by simply restoring \( \xi^+ \) or \( \xi^- \) which correspond to collision states.

To accomplish this equivalence we will need to show

**LEMMA 1.** \( \mathcal{H}_+ \) or \( \mathcal{H}_- \) can be mapped isometrically onto the Lobachevsky disc \( w_1^2 + w_2^2 < 1 \) with the metric \( ds^2 = 4(dw_1^2 + dw_2^2)/(1 - w_1^2 - w_2^2)^2 \).

To prove the above theorem and lemma we will first show that the Lorentz metric \( ds^2 = -d\xi_0^2 + d\xi_1^2 + d\xi_2^2 \) is a Riemannian metric (i.e., positive definite) when restricted to \( \mathcal{H}_\pm \). This is easily proven by showing that the operation \( \langle \alpha, \beta \rangle = -\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 \) defines an inner product for all vectors \( \alpha, \beta \in T(\mathcal{H}_\pm) \) – the tangent bundle of \( \mathcal{H}_\pm \). To this end we can represent \( \mathcal{H}_\pm \) parametrically as

\[
\xi = (\xi_0, \xi_1, \xi_2) = (\pm \cosh u, \sinh u \cos v, \sinh u \sin v). \quad (3)
\]