ERROR PROPAGATION IN THE NUMERICAL SOLUTIONS OF THE DIFFERENTIAL EQUATIONS OF ORBITAL MECHANICS

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(Received April, 1981; accepted September, 1981)

Abstract. The relationship between the eigenvalues of the linearized differential equations of orbital mechanics and the stability characteristics of numerical methods is presented. It is shown that the Cowell, Encke, and Encke formulation with an independent variable related to the eccentric anomaly all have a real positive eigenvalue when linearized about the initial conditions. The real positive eigenvalue causes an amplification of the error of the solution when used in conjunction with a numerical integration method. In contrast an element formulation has zero eigenvalues and is numerically stable.

1. Introduction

This paper will consider the propagation of errors in the differential equations of orbital motion when numerical methods are used to solve them. For the past several years, the technique used by the developers of numerical methods, for example, Lambert (1973), Shampine and Gordon (1975), and Bettis (1977), for determining stability characteristics has been that of applying the numerical algorithm to a single linear differential equation

\[ \frac{dy}{dx} = \lambda y, \quad y(x_0) = y_0 \]  

(1)

and then determining how an error in \( y \) propagates over a single step. The result of this application is the numerical solution for \( y \) at the \( (n+1) \) step; that is,

\[ y_{n+1} = P(\lambda h)y_n, \]  

(2)

where \( y_n \) is the value of \( y \) prior to the step. \( P(\lambda h) \) is a function of the method and the product of the step size \( h = x_{n+1} - x_n \) times the complex eigenvalue \( \lambda \). The error has an identical differential equation and numerical solution as in Equations (1) and (2). In order for the error not to increase, it is necessary that

\[ P(\lambda h) \leq 1. \]  

(3)

The function \( P(\lambda h) \) may be regarded as the amplification factor of the error. A plot is made of \( P(\lambda h) \) in the complex plane, and from this plot the applicability of the method to a particular system of ordinary differential equations may be determined. The question arises as to the validity of applying the results of this technique, which uses only a single linear differential equation, to a system of nonlinear differential equations. This question may be answered as follows: (1) If the nonlinear differential equations are linearized, then the linearized equations can be used with reasonable
safety to approximate the stability characteristics of the nonlinear system of equations over one integration step; (2) Most systems of linear differential equations can be diagonalized by similarity transformations, that maintain the eigenvalues, into \( n \) equations of the type in Equation (1). From an error analysis viewpoint, if such a linearization and diagonalization can be done, only eigenvalues with positive real components will be considered; that is, any eigenvalue that produces error amplification over a single step. (One should keep in mind that error amplification during a numerical solution may also occur due to eigenvalues having negative real parts, but the integration step length can be reduced to avoid this problem).

The relation between the eigenvalues of each of four different formulations of the two-body problem and the stability characteristics of the numerical method will be presented. Three of the formulations, all in Cartesian coordinates, are the Cowell, the Encke, and the Encke-\( \beta \). The Encke-\( \beta \) is similar to the Encke but has an independent variable \( \beta \) which is related to time by

\[
\frac{dt}{d\beta} = (\text{constant}) \cdot r,
\]

where \( r \) is the magnitude of the position vector and \( \beta \) is proportional to the eccentric anomaly. The fourth formulation is based upon the integrals of motion or elements.

2. Stability of Linearized Systems

The criterion for local stability of the solution of any system of nonlinear ordinary differential equations is that the solution to the corresponding linearized system of differential equations should be stable. The determination of the eigenvalues of the linearized system is necessary for stability analysis. The procedure is outlined in this section and is taken from page 128 of Shampine and Gordon (1975). Consider the nonlinear system of ordinary differential equations.

\[
\frac{dy}{dx} = f(y, x), \quad y_0 = y(x_0), \tag{4}
\]

where \( y \) is a vector \((y_1, y_2 \ldots y_n)\), and \( x \) is the independent variable.

The nonlinear system (4) may be approximated near a point \( x_i \) by the linear system of differential equations.

\[
\frac{dy}{dx} = Ay + g(x), \tag{5}
\]

where \( A \) is the constant \( n \) by \( n \) Jacobian matrix

\[
A = \left[ \frac{\partial f}{\partial y} \right]_{x_i}.
\]

Equation (5) is found by expanding the right hand side of (4) in a Taylor series about