PERIODIC SOLUTIONS OF CIRCULAR-ELLIPTIC TYPE
IN THE PLANAR N-BODY PROBLEM

R. F. ARENSTORF
Dept. of Mathematics, Vanderbilt University, Nashville, Tenn., U.S.A.

(Received 22 September, 1977)

Abstract. Let \( n \geq 2 \) mass points with arbitrary masses move circularly on a rotating straight-line
central-configuration; i.e. on a particular solution of relative equilibrium of the \( n \)-body problem.
Replacing one of the mass points by a close pair of mass points (with mass conservation) we show that
the resulting \( N \)-body problem \((N = n + 1)\) has solutions, which are periodic in a rotating coordinate
system and describe precessing nearly elliptic motion of the binary and nearly circular collinear motion
of its center of mass and the other bodies; assuming that also the mass ratio of the binary is small.

1. Introduction

We consider the planar \( N \)-body problem for \( N \geq 3 \) mass points with masses \( m_k > 0 \)
and inertial position vectors \( q_k, \ (k = 1, \ldots, N) \). The equations of motion (with
\( = d/dt \) ) are

\[
m_k \ddot{q}_k = \nabla q_k V_N, \quad (k = 1, \ldots, N), \quad V_N = \sum_{1 \leq j < k \leq N} m_j m_k \left| q_j - q_k \right|^{-1}.
\]  

(1.1)

We shall identify the plane of motion \( \mathbb{E}^2 \) with the ordinary complex plane \( \mathbb{C} \), and we
shall use notations in \( \mathbb{E}^2 \) and \( \mathbb{C} \) simultaneously, for example

\[
p \cdot q = \text{Re} \ p \bar{q}; \ \nabla_x H = H_{x_1} + i H_{x_2}, \quad \text{if} \quad x = x_1 + i x_2 = (x_1, x_2) \in \mathbb{E}^2;
\]

for the scalar product of \( p \) and \( q \); respectively for the gradient and the partial deriva-
tives of \( H(x) \).

If \( |q_N - q_{N-1}| \) say, is permanently small compared to the other distances, this pair
of bodies forms a binary whose equations of motion can be approximated by the
2-body problem. The equations of motion for the remaining \( N - 2 \) bodies together
with a fictitious body of mass \( m_n^* = m_N + m_n, (n = N - 1) \) at the center of mass \( q^* \) of
the binary, can be approximated by the corresponding \( n \)-body problem. From this
latter problem we take the known circular motion of collinear relative equilibrium; i.e.

\[
q_k = e_k e^{it}, \quad (k = 1, \ldots, n - 1), \quad q^* = e_n e^{it},
\]

(1.2)

where the real constants \( e_1, \ldots, e_n \) denote the positions in \( \mathbb{C} \) of a collinear central
configuration; i.e. \( (e_1, \ldots, e_n) \) is a solution of the system of equations

\[
\sum_{j=1}^{n} m_j^* F(e_j - e_k) = -e_k \text{ (real)}, \quad (k = 1, \ldots, n); \quad F(z) = z \left| z \right|^{-3}; \quad (1.3)
\]
belonging to the masses

\[ m_j^* = m_j, \quad (j = 1, \ldots, n - 1), \quad m_n^* = m_n + m_N, \quad (n = N - 1). \quad (1.4) \]

It is known that the number of solutions of (1.3) is always positive and finite; in particular, it is \( n! \), when \( m_1^*, \ldots, m_n^* \) have different values. We shall keep these \( n \) masses fixed throughout the following work, only the mass ratio \( \mu = m_N/m_n^* \) of the binary will be chosen sufficiently small (as specified later).

From the former 2-body problem we take an elliptic motion with small semi-major axis \( a^* \) and arbitrary eccentricity \( \varepsilon \) in \((0, 1)\) and such that its inertial period \( T \) is commensurable with that of the \( n \)-body motion in (1.2); i.e.

\[ T |k_0| = 2\pi m_0, \quad (k_0 \neq 0), \quad a^* = |m_n^*(m_0/k_0)^2|^{1/3} \quad (1.5) \]

with relatively prime integers \( k_0 \) and \( m_0 \); so that this motion becomes periodic (closing after \( k_0 - m_0 \) revolutions of the binary) with period \( T_0 = 2\pi m_0 \) in a rotating coordinate system in which the other \( n \) bodies described above remain stationary. Clearly the preceding description gives only a heuristic approximation to some expected periodic solutions of the \( N \)-body problem.

In this paper we present a proof of the existence of 1-parameter families of relative periodic solutions of the actual \( N \)-body problem (1.1) for any \( N \geq 3 \), which are close to the approximate 'solution' described above. These solutions are periodic in a uniformly rotating coordinate system with periods close to \( 2\pi m_0 \); the family parameter is the approximate eccentricity \( \varepsilon \) and the different families correspond to the admissible rational numbers \( m_0/k_0 \). In the inertial coordinate system they describe precessing nearly 'elliptic' motion for the close binary \((q_n, q_N)\) and nearly 'circular' motion for its center of mass and the other bodies. We shall call such motions of 'circular-elliptic type'. They exist according to the following proof (at least) when the mass ratio of the binary and the constants \( k_0, m_0, \varepsilon \) of its approximating Keplerian motion are chosen such that

\[ 0 \leq \mu \leq c a^* \leq c^* \quad \text{and} \quad 0 < \varepsilon_1 \leq \varepsilon \leq \varepsilon_2 < 1, \quad (1.6) \]

where \( c \) and \( c^* \) are constructable functions of \( \varepsilon_1, \varepsilon_2 \) and \( m_0 \), under the assumption when \( N > 3 \), that the given masses \( m_1^*, \ldots, m_n^* \) avoid a certain set of measure zero, described later by Equation (9.13). Thus \( m_0 \) is almost arbitrary but implies the ranges of \( \mu \) and \( k_0 \). It would be desirable to remove the restriction on \( \mu \).

The present result extends the recent work of Arenstorf and Bozemann (1978) from the restricted \((\mu = 0)\) to the full \( N \)-body problem. It also generalizes to \( N > 3 \) an earlier result of the author (1968), dealing with the lunar theory \((N = 3)\) even without restrictions on the three masses. The existence of relative periodic solutions analogous to the ones described above, but of 'circular-circular type'; i.e. with a nearly circular binary, has been established by Perron (1937a, b). For \( N = 4 \) Crandall (1967) has shown the existence of relative periodic solutions of 'circular-circular type' such that a close binary moves nearly circular about one of the vertices of an equilateral triangle,