A RELATION IN FAMILIES OF PERIODIC SOLUTIONS

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Abstract. We show the existence of a general relation between the parameters of periodic solutions in dynamical systems with ignorable coordinates. In particular, for time-independent systems with an axis of symmetry, the relation takes the form $\frac{\partial T}{\partial A} = -\frac{\partial \Phi}{\partial E}$, where $T$ is the period, $A$ is the angular momentum, $\Phi$ is the angle through which the system has rotated after one period, and $E$ is the energy.

1. The General Relation

We wish to call attention to a curious relation which is obeyed by families of periodic solutions in dynamical systems. This relation does not seem to have been noticed so far, although it is rather simple.

We begin by recalling some classical notions (Whittaker, 1937, Section 38; Goldstein, 1956, Section 7.2). Consider a dynamical system with $n$ degrees of freedom, defined by a Lagrangian

$$L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$$

and suppose that there are $k$ ignorable coordinates. We may choose them to be $q_1, \ldots, q_k$, and the Lagrangian reduces to

$$L(q_{k+1}, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t).$$

There are then $k$ integrals of the motion

$$\frac{\partial L}{\partial \dot{q}_i} = \beta_i, \quad (i = 1, \ldots, k)$$

and the system can be reduced to $n-k$ degrees of freedom by the process of ignorance of coordinates. A new function, called the Routhian, is introduced

$$R = \sum_{i=1}^{k} \beta_i \dot{q}_i - L.$$  \hspace{1cm} (4)

The $k$ Equations (3) are solved for $\dot{q}_1, \ldots, \dot{q}_k$ and the resulting expressions are substituted in (4), so that $R$ takes the form

$$R(q_{k+1}, \ldots, q_n, \dot{q}_{k+1}, \ldots, \dot{q}_n, t, \beta_1, \ldots, \beta_k).$$ \hspace{1cm} (5)

As is easily shown, $R$ is then the Lagrangian of a reduced system with $n-k$ coordinates $q_{k+1}, \ldots, q_n$. The quantities $\beta_1, \ldots, \beta_k$ appear as parameters in this reduced system.
Once a solution of the reduced system has been found, the corresponding solutions of the original system are obtained by computing the ignorable coordinates from

\[ q_t = \int \frac{\partial R}{\partial \beta_i} \, dt, \quad (i = 1, \ldots, k). \]  

(6)

Note that \( k \) integration constants will appear as arbitrary additive constants in the \( q_t \).

We shall assume now that the Lagrangian (2) is periodic with respect to time, and we take the period equal to 1 for convenience

\[ L(q_{k+1}, \ldots, \dot{q}_n, t + 1) = L(q_{k+1}, \ldots, \dot{q}_n, t). \]  

(7)

The Routhian (5) is then also periodic with period 1. From now on we restrict our attention to periodic solutions of the reduced system, i.e. solutions such that the reduced variables and their derivatives come back to their initial values after one period

\[ q_i(1) = q_i(0), \quad \dot{q}_i(1) = \dot{q}_i(0), \quad (i = k + 1, \ldots, n). \]  

(8)

For given values of \( \beta_1, \ldots, \beta_k \), we may expect such periodic solutions to exist in general, because (8) represents a system of \( 2(n-k) \) conditions on the \( 2(n-k) \) initial values.

We consider now a particular periodic solution of the reduced system

\[ q_i(t), \quad (i = k + 1, \ldots, n) \]  

(9)

and the corresponding solutions of the original problem. According to (6), after one period each ignorable coordinate \( q_t \) has increased by a quantity \( T_i \), given by

\[ T_i = q_i(1) - q_i(0) = \int_0^1 \frac{\partial R}{\partial \beta_i} \, dt, \quad (i = 1, \ldots, k). \]  

(10)

The \( T_i \) have no reason to vanish in general; thus, the ignorable coordinates do not come back to their initial values, and the solutions of the original problem are not periodic in general.

The quantities \( T_i \), which we shall call generalized periods, do not depend on the integration constants in (6). They do not depend either on the origin of time, i.e., there is more generally

\[ T_i = q_i(t + 1) - q_i(t). \]  

(11)

Thus, the generalized periods \( T_i \) can be considered as intrinsic parameters of the periodic solution. They often have a simple physical meaning, as we shall see below.

Another intrinsic parameter of the periodic orbit is the Lagrangian action (Synge, 1960, Section 65), computed over one period

\[ S = \int_0^1 R \, dt. \]  

(12)