Analytical Determination of the Measure of Stability of Triple Stellar Systems

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Abstract. An analytical stability criterion is given by the limiting value of the ratio of the semi-major axes of the outer star and inner binary and Harrington's (1972) numerically obtained results are slightly modified for planar systems in direct motion.

Introduction

The angular momentum (c) and the total energy (H) of a classical triple system are computed using two-body approximations. Then, this result is combined with the critical value of the quantity (c²H) obtained from the surfaces of zero velocity of the general planar problem of three bodies. This is followed by the computation of the critical semi-major axis ratio of the outer star and of the binary. Possible extensions of the theory are also offered and comparisons are given with Harrington's (1972) results and with actual triple systems.

1. Two-Body Approximations

The masses, semi-major axis, and eccentricity of the binary are m₁, m₂, a₁, and e₁. The corresponding quantities for the outside star are m₃, a₂, and e₂. The total mass of the system is M, that of the binary is µ and the gravitational constant is G. The Keplerian energy and angular momentum are

\[ H = -\frac{G}{2} \left( \frac{m_1 m_2}{a_1} + \frac{m_3 \mu}{a_2} \right), \]

and

\[ c = G^{1/2} \left[ m_1 m_2 [a_1 (1 - e_1^2)]/\mu \right]^{1/2} + m_3 \mu [a_2 (1 - e_2^2)]/M \]^{1/2}. \]

The pertinent parameter controlling the stability through its effect on the surfaces of zero velocity is \( s = c² |H|/G² \), which may be obtained from Equations (1) and (2) as

\[ s = \frac{a_2}{a_1} M_1 E_2 + 2 \left( \frac{a_2}{a_1} \right)^{1/2} M_2 (E_1 E_2)^{1/2} + M_5 E_1 + \]

\[ + \frac{a_1}{a_2} M_3 E_1 + 2 \left( \frac{a_1}{a_2} \right)^{1/2} M_4 (E_1 E_2)^{1/2} + M_6 E_2, \]
where the mass-factors $M_i$ and the eccentricity-factors $E_i$ are given by

$$
M_1 = m_1 m_2 m_3 \mu^2 / M, \quad M_2 = m_1^2 m_2 m_3 (\mu / M)^{1/2}, \\
M_3 = m_2^2 m_3, \quad M_4 = m_1 m_2^2 m_3 (\mu / M)^{1/2}, \\
M_5 = m_1^2 m_2^2 \mu, \quad M_6 = m_3^3 \mu^3 / M, \quad E_1 = (1 - e_1^2) / 2.
$$

Using Equation (3) the stability parameter $s$ may be computed for an actual triple system. On the other hand, the ratio $a_2/a_1$ may be evaluated if the masses, eccentricities and the value of $s$ are given.

2. Critical Value of the Stability Parameter

The surfaces of zero velocity allow exchange of bodies at a critical value of $s$. As long as the actual value of this quantity $s_a$ is larger than the critical value of $s_c$ such exchanges can not occur and the triple configuration is stable in this sense. Consequently, a measure of stability may be introduced as

$$S = \frac{s_a - s_c}{s_c}.$$  

For large values of $S$ the triple system under investigation is far removed from possible exchanges and, therefore, it has a large stability margin. If $S=0$ the system is on the borderline between stability and instability. When $S$ is negative, exchange of the participating stars may occur. Note that our stability condition is sufficient but not necessary since exchange might or might not occur when $S<0$ but it certainly will not happen when $S>0$. (Furthermore, there is nothing in the theory of zero velocity surfaces that would prevent escape since the surfaces are always open to infinity.)

The critical value of $c^2H$ depends only on the value of the participating masses, while the actual value is computed from Equation (3) and, therefore, it depends, in addition to the masses, on the semi-major axes and on the eccentricities.

The critical value for equal masses ($m_1 = m_2 = m_3 = m$) is $s_c = (24/4)m^5$. This value may be obtained from recent results by Marchal (1976), Marchal and Saari (1975), Zare (1976), and Bozis (1976). For any values of masses $s_c$ may be also computed, see for instance Zare (1976).

3. Computation of the Ratio $a_2/a_1$ for Stability

When the above critical value is substituted for $s$ in Equation (3), we obtain an equation for the lowest value of $a_2/a_1$ which allows stable motion. For equal masses and zero eccentricities Equation (3) gives

$$6.25 = \frac{2}{3} \frac{a_2}{a_1} + \left( \frac{2}{3} \frac{a_2}{a_1} \right)^{1/2} + 2 \left( \frac{2}{3} \frac{a_1}{a_2} \right)^{1/2} + \frac{1}{2} \frac{a_1}{a_2} + 19/12. \quad (4)$$

The solution of this 4th order equation for $a_2/a_1$ is very close to 3.2, therefore, we