A TRANSFORMATION OF THE TWO-BODY PROBLEM

VICTOR R. BOND
NASA-Johnson Space Center, Houston, Tex., U.S.A.

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ABSTRACT. A transformation of the differential equations of motion of the two-body problem in the spherical coordinates to oscillator form is derived. It is shown that the independent variable transformation \( \frac{dt}{ds} = r^2 \) is a transformation which makes the oscillator form possible.

1. INTRODUCTION

It was shown by Szebehely and Bond (1983) that the differential equations of motion for the perturbed problem of two bodies could be transformed into oscillator form when the motion was constrained to the orbital plane. This paper showed that the transformation was obtained for the distance transformation, \( r = F(\rho) \) where \( r \) and \( \rho \) are the old and new distance variables, and the independent variable transformation \( \frac{dt}{ds} = g(r) \). A differential equation was derived that related the transformations of the independent and dependent variables. All transformations which satisfy this relation result in harmonic oscillators.

Any transformation of the perturbed two-body problem which is limited to the planar case is restrictive even if perturbations are included since in most cases, perturbations produce deviations which are normal to the orbital plane. For practical application of this theory it is necessary that the transformation be generalized to three dimensions. In the following we obtain the transformation by considering the problem of two bodies only - relaxing the requirement that perturbations be included in the transformation. The perturbations can be included by variation of parameters technique.

This paper will extend the linearization to the three-dimensional case by starting with the classical differential equations of motion in spherical coordinates. The differential equations for the distance, latitude, and longitude \( (r, \theta, \phi) \) will be transformed into differential equations for the new variables, corresponding to distance, latitude, and longitude \( (\rho, \lambda, \alpha) \) which have the oscillator form

\[
\rho'' + c^2 \rho = \mu, \quad \lambda'' + c^2 \lambda = 0, \quad \alpha'' + c^2 \alpha = 0,
\]

where \( c \) is the total angular momentum and \( \mu \) is the product of the gravitational constant and the sum of the masses of the two bodies. It is further shown that the independent variable transformation which produces the above
differential equations is
\[ t' = \frac{dt}{ds} = g(r) = r^2, \]
which means that the new independent variable \( s \) is proportional to the true anomaly. For example, see Szebehely and Bond (1983).

The transformations between the new and old dependent variables,
\[ r = F(\rho), \quad \theta = f(\lambda), \quad \psi = \psi(\alpha), \]
will be shown.

2. TRANSFORMATION TO OSCILLATOR FORM

The equations of motion are given in spherical coordinates by
\[
\begin{align*}
r' &= \frac{c^2}{c^3} - \frac{\dot{\mu}}{r^2}, \\
\frac{d}{dt}(r^2 \dot{\theta}) &= -\frac{p^2}{r^2} \frac{\sin \theta}{\cos^3 \theta}, \\
\ddot{\psi} &= \frac{p_{\psi}}{r^2 \cos^2 \theta},
\end{align*}
\]
where \( r, \theta, \psi \) are the distance, latitude, and longitude. For convenience, we refer to these three equations as the distance, latitude, and longitude differential equations, respectively. Equation (3) represents an integral of the motion,
\[ p_{\psi} = \text{a constant} \]
which is the component of the angular momentum normal to the equatorial plane or other reference plane. There are two other integrals of the motion
\[ c^2 = r^4 (\dot{\theta}^2 + \dot{\psi}^2 \cos^2 \theta) \]
which is the total angular momentum squared, and
\[ h = \frac{1}{2} (\dot{r}^2 + \frac{c^2}{r^2}) - \frac{\mu}{r} \]
which is the energy.

We note that Equation (1) is identical to the differential equation for the distance for the planar case which was considered by Szebehely and Bond (1983). It was shown there that by transforming the independent variable from \( t \) to \( s \) using a generalization of Sundman's (1912) transformation
\[ \frac{dt}{ds} = g(r) \]
and transforming the distance \( r \) to \( \rho \) according to
\[ r = F(\rho), \]