VIRTUAL SINGULARITIES IN
THE ARTIFICIAL SATELLITE THEORY

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Abstract. When the coordinate system used in perturbation theory presents a geometrical singularity and when the perturbation technique fails to take account of this, the solution developed may present singularities which are no longer easily explained by purely geometrical means. These singularities have been called virtual singularities by Deprit and Rom (1970). We propose to demonstrate that virtual singularities can in general be avoided by the use of Lie transforms. In general, it is sufficient to recognize that the original Hamiltonian function presents the d'Alembert characteristic with respect to pairs of action-angle variables and that the averaging operations preserve this characteristic. We then apply this criterion to the artificial satellite theory (for small to moderate eccentricity) showing that all of three possible virtual singularities can be avoided at the same time. A new set of elliptic elements, well suited to the problem at hand, is introduced.

1. Introduction

In Celestial Mechanics as well as in nonlinear mechanics, problems to be solved by perturbation theory are usually expressed in a polar-like set of coordinates where the radius vector \( r \) changes slowly with time while the angular variable \( \phi \) is almost a linear function of the time. It is in this setting that the characteristic behaviour of the system is best made apparent: a fast oscillation whose phase and amplitude is slowly varying.

The transformation which brings about these polar coordinates is singular for \( r = 0 \) and the angle \( \phi \) is then undetermined. This is a purely geometrical singularity and we know that to avoid analytical or numerical difficulties we can always come back to a Cartesian set of coordinates: \( x = r \sin \phi \), \( y = r \cos \phi \) when dealing with small values of \( r \).

The situation is trivial at this stage but becomes more involved when a transformation from \( (r, \phi) \) to a new set of polar-like coordinates, say \( (s, \psi) \) is performed in the implementation of the perturbation theory. For instance the point \( r = 0 \) may not be identical to the point \( s = 0 \) and thus the transformation from \( (s, \psi) \) to a Cartesian-like set of coordinates regularizing the singularity \( r = 0 \) may not be obvious. Also some transformations, for instance

\[
\begin{align*}
s &= r + \varepsilon \cos \phi \\
\psi &= \phi
\end{align*}
\]

(1.1)
do not make sense geometrically and should be avoided.

Thus one ought to be careful in defining transformations of polar-like coordinates.
because they may introduce in the equations what Deprit and Rom (1970) call virtual singularities, i.e. singularities which in the initial stage of the problem are known to be purely geometrical and do not present any difficulties but which in a later stage may have affected implicitly or explicitly so many parts of the problem that it is difficult to account for them in a purely geometrical manner.

Singularities for small eccentricities appearing in the literal developments of Lunar Theory were one of the main difficulties encountered by Barton (1967). It was so severe that it made him abandon his project of reconstructing Delaunay's Theory by using the so-called Von Zeipel method on a computer. This difficulty was later completely overcome by the use of Lie transforms (Deprit et al., 1971b).

Singularities for small eccentricities and small inclinations are present in Brouwer's Theory of an artificial satellite (Brouwer, 1959). These singularities present not only analytical problems (what is the meaning of the theory when the eccentricity is zero?) but numerical problems as well. The truncation errors of the solution, which increase for small eccentricities, cannot be justified by the dynamics of the problem. Among the suggestions made to eliminate these virtual singularities, we ought to mention especially those of Lyddane (1963) and Deprit and Rom (1970).

There is in the problem of an artificial satellite another possible virtual singularity which to our knowledge has not yet been investigated. It appears when the inclination angle takes the value \( \pi \) and it is symmetric with the singularity for small inclination.

In this paper, we plan to clarify the notion of virtual singularities and to show how they can be avoided in Perturbation Theory by the use of Lie transforms.

Then we apply this technique to the problem of an artificial satellite and conclude that the three possible virtual singularities (small eccentricity, small inclination, inclination close to 180°) can be eliminated together.

In the last section, we introduce a set of elliptic elements which makes apparent the geometrical character of those singularities. Because of its symmetry and simplicity, this set should be well adapted to the implementation of an artificial satellite theory along the lines of Deprit's theory (Deprit and Rom, 1970).

2. Perturbation Theory and Virtual Singularities

Basically a perturbation theory expresses an osculating set of elements, say \( x = \{x_1, ..., x_n\} \) which determine the phase state with respect to an averaged set of elements, say \( x = \{\bar{x}_1, ..., \bar{x}_n\} \) which are known functions of the time. These expressions define a transformation from the 'coordinates' \( x_i \) to the 'coordinates' \( \bar{x}_i \) which is close to the identity; i.e. we have

\[
x_i = \bar{x}_i + \epsilon x_i(\bar{x}), \tag{2.1}
\]

where \( \epsilon \) is a small quantity which usually is an external parameter, like the oblateness of the Earth in the problem of an artificial satellite, but which may be a scale