THE ELIMINATION OF THE PARALLAX IN
SATELLITE THEORY

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Abstract. When the perturbation affecting a Keplerian motion is proportional to \( r^{-n} \) \( (n \geq 3) \), a canonical transformation of Lie type will convert the system into one in which the perturbation is proportional to \( r^{-2} \). Because it removes parallactic factors, the transformation is called the elimination of the parallax.

In the main problem for the theory of artificial satellites, the elimination of the parallax has been conducted by computer to order 4. The first order in the reduced system may now be integrated in closed form, thereby revealing the fundamental property of the first-order intermediary orbits in line with Newton's Proposition XLIV.

Extension beyond order 1 leads to identify a new class of intermediaries for the main problem in nodal coordinates, namely the radial intermediaries.

The technique of smoothing a perturbation prior to normalizing the perturbed Keplerian system, of which the elimination of the parallax is an instance, is applied to derive the intermediaries in nodal coordinates proposed by Sterne, Garfinkel, Cid-Palacios and Aksnes, and to find the canonical diffeomorphisms which relate them to one another and to the radial intermediaries.

1. Introduction

So much having been published about the main problem in the theory of an artificial satellite, it may be taken as axiomatic that a claim of originality may (and probably will) be met by a statement of the opposite. With all this wealth, empty spaces nevertheless exist where an author can contribute more than an inadvertent duplication or a personal derivation of results already in print. We believe the present communication falls into one of the cracks. The elimination of the parallax is a technique by which a Keplerian system in a central field of force in \( r^{-n} \) \( (n \geq 3) \) is converted into a quasi-Keplerian system with a variable angular momentum. To our knowledge, this technique is new to perturbation theories in non-linear and celestial mechanics.

In the theory of artificial satellites, the equations of motion are generally solved analytically in two steps. A canonical transformation eliminates the short period terms depending on the mean anomaly. Then the averaged equations are solved either by numerical integration, as in the semi-analytical theories of Lidov and Cefola (1974) or Wu Lian-Da et al. (1978), or by analytical normalization, a second canonical transformation being performed to remove the long-period terms in the argument of perigee and the longitude of the ascending node. Both methods tend generally to underestimate the amount of work required in the course of averaging over the mean anomaly. Yet the practical difficulties met in computing average elements from initial conditions and in evaluating multivariate Fourier series have discouraged engineers from programming analytical solutions for predicting and improving...
positions of satellites in real time. In this context, the elimination of the parallax presents real merits. In a kind of feasibility study to justify a programming effort of long duration, the parallax has been eliminated from the main problem to order 4. Even though they have no realistic use, the results are reported here to let the reader be the judge, at the onset, of our claims to simplicity. Soon after the present manuscript appears in print, there will come a report on what has been accomplished by removing the parallax in a gravity field including the zonal harmonics from $J_2$ to $J_7$. It will show how radically the elimination has disentangled the problem.

Undeniably the method imposes the conventional algorithms to start with yet another canonical transformation. However, the procedure amounts to factoring the elimination of the short-period terms into the product of two transformations, each one definitely simpler than the product. The elimination of the parallax removes those short-period variations due to a torsion exercised on the apsidal frame by the perturbations in $r^{-n}$ ($n \geq 3$) and in the argument of latitude. Left in the system are the short-period variations resulting from irregularities in the equation of the centre. The factoring is reminiscent of the parceling of terms made in lunar theory by de Pontécoulant.

Computer programs are in the making to eliminate the parallax in the theory of artificial satellites; they are built by stepwise refinements, each extension closely collating its results with developments obtained by various authors. How else could one guard the computer programs from slipping into error? While this sustained vigilance stimulates to assimilate anterior contributions into the present scheme, it may (and will) trap some readers into commenting that the 'last word has been said long ago in the theory of artificial satellites'. To the contrary, we believe that, stripped of its non-essential complications, the main problem is now amenable to further investigations into its characteristic structures.

The computational complexities induced by the short-period fluctuations have been paid a great deal of attention since they were uncovered by Kozai (1962) while he extended to order 2 the first-order solution developed by Brouwer (1959). One way of mitigating the difficulty consists in breaking the main problem in two parts, a dominant term made of a separable Hamiltonian called the intermediary and a remainder to be regarded as a perturbation affecting the intermediary (Sterne, 1960). In spheroidal coordinates (Vinti, 1959; Kislik, 1961), the intermediaries contain the whole first-order part of the main problem. Apparently this is not the case for the intermediaries in polar coordinates (Sterne, 1957; Garfinkel, 1958, 1959; Cid–Palacios, 1969; Aksnes, 1965, 1966). The deficiency, however, is remedied by reformulating the concept of intermediary. Given a Hamiltonian in the map $(q, Q)$, a natural intermediary for $H$ will be defined as a pair $(\phi, J)$ made of a canonical transformation $\phi : (q, Q) \rightarrow (p, P)$ and of a separable Hamiltonian $J$ in the chart $(p, P)$ such that the perturbation $H[q(p, P), Q(p, P)] - J(p, P)$ be of second order. For each intermediary in the sense of Sterne–Garfinkel, there exists an infinitesimal contact transformation which makes it a natural intermediary. This new concept