PARTITIONING PROJECTIVE GEOMETRIES INTO SEGRE VARIETIES

In ricordo di Giuseppe Tallini

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Using suitable subgroups of Singer cyclic groups we prove some properties of regular spreads and Segre varieties, which in turn yield a necessary and sufficient condition for partitioning a finite projective space into such varieties.

Assume $n = m \cdot k$ with $1 < m, k < n$. With the notation of [2] let $S_{m,k}$ be a Segre variety with indices $m$ and $k$ in $PG(n-1, q)$. Denote by $R_{m-1}$ the family of $(m-1)$-dimensional projective subspaces lying on $S_{m,k}$. L.R.A. Casse and C.M. O'Keefe in [2] call $R_{m-1}$ an $(m-1)$-regulus of rank $k-1$ in $PG(n-1, q)$. In the same paper they define an $(m-1)$-spread $F$ of $PG(n-1, q)$ to be regular (of rank $k-1$) if, for any $(k-1)$-dimensional subspace $A$ of $PG(n-1, q)$ meeting each member of $F$ in at most one point, the members of $F$ actually meeting $A$ form an $(m-1)$-regulus of rank $k-1$.

In the paper [1] by the same authors it is shown that the above concept of a regular spread is equivalent to the following generalization of the notion of a linear elliptic congruence, see also §27 in [4]. Let $\Lambda_0, \Lambda_1, \ldots, \Lambda_{m-1}$ be pairwise skew $(k-1)$-dimensional subspaces of $PG(n-1, q^m)$ which span $PG(n-1, q^m)$ and are conjugate over $GF(q)$ with respect to $GF(q^m)$ (that is $\Lambda_i = \Lambda_i^q$, where $\sigma$ denotes the collineation of $PG(n-1, q^m)$ arising from the Frobenius automorphism $GF(q^m) \rightarrow GF(q^m), x \mapsto x^q$). For any point $P_0 \in \Lambda_0$ set $P_i = P_0^\sigma^i$. Then $P_0, \ldots, P_{m-1}$ are conjugate points over $GF(q)$ with respect to $GF(q^m)$. Furthermore, $P_0, \ldots, P_{m-1}$ span an $(m-1)$-dimensional subspace $\Pi$ of $PG(n-1, q^m)$ intersecting $PG(n-1, q)$ in an $(m-1)$-dimensional subspace $\Pi$. The set of all such subspaces $\Pi$ is a regular $(m-1)$-spread in $PG(n-1, q)$.

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Let $\hat{S} = \langle \varphi \rangle$ be a Singer cyclic group in $PGL(n, q)$, that is a cyclic subgroup of $PGL(n, q)$ acting regularly on the points of the underlying projective geometry $PG(n - 1, q)$. Let $m$ be a non-trivial proper divisor of $n$ and set $k = n/m$. Denoting by $\hat{T}$ the subgroup of order $(q^m - 1)/(q - 1)$ of $\hat{S}$, we have $\hat{T} = \langle \varphi^{(q^m - 1)/(q - 1)} \rangle$. As $\hat{T}$ acts semiregularly on $PG(n - 1, q)$, each $\hat{T}$-orbit has length $(q^m - 1)/(q - 1)$. Setting $s = 1 + q^m + \ldots + q^{m(k-1)}$ we denote by $\Pi_0, \Pi_1, \ldots, \Pi_{s-1}$ the $\hat{T}$-orbits on $PG(n - 1, q)$.

**Proposition 1.** Setting $\mathcal{F} = \{\Pi_0, \Pi_1, \ldots, \Pi_{s-1}\}$ we have that $\mathcal{F}$ is an $(m - 1)$-spread of $PG(n - 1, q)$ which is regular in the above sense.

**Proof:** The fact that $\Pi_0, \Pi_1, \ldots, \Pi_{s-1}$ form a spread of of $PG(n - 1, q)$ into $(m - 1)$-dimensional subspaces is well known, see for instance the discussion in §4.2 of [3].

Assume now $\varphi$ is represented by the matrix $U \in GL(n, q)$. The matrix $U$ is conjugate in $GL(n, q)$ to $D = \text{diag}(\zeta, \zeta^q, \ldots, \zeta^{q^{n-1}})$ for some primitive element $\zeta \in GF(q^n)$, see again [3]. Therefore $U^{(q^{n-1})/(q^{m-1})}$ is conjugate in $GL(n, q^m)$ to $E = D^{(q^{n-1})/(q^{m-1})} = \text{diag}(\xi_1, \xi_2, \ldots, \xi_k)$ where $\xi_i = \zeta^{(q^{n-1})/(q_{m-1})}$. Note that since $\xi \in GF(q^m)$ we have that $U^{(q^{n-1})/(q^{m-1})}$ is also conjugate to $E$ in $GL(n, q^m)$. For $j = 0, 1, \ldots, m - 1$ the $k$-dimensional eigenspace of $U^{(q^{n-1})/(q^{m-1})}$ with respect to the eigenvalue $\xi^j$ yields a $(k - 1)$-dimensional projective subspace $\Lambda_j$ of $PG(n - 1, q^m)$.

Clearly $\Lambda_0, \Lambda_1, \ldots, \Lambda_{m-1}$ generate $PG(n - 1, q^m)$ and are conjugate over $GF(q)$ with respect to $GF(q^m)$.

Consider $PG(n - 1, q)$ as embedded in $PG(n - 1, q^m)$ and let $\sigma$ denote the collineation of $PG(n - 1, q^m)$ arising from the field-automorphism $GF(q^m) \rightarrow GF(q^m)$, $x \mapsto x^q$. Each subspace $\Pi_i$ is contained in a unique $(m - 1)$-dimensional subspace $\overline{\Pi}_i$ in $PG(n - 1, q^m)$ and we have $\overline{\Pi}_i^\sigma = \overline{\Pi}_i$, $i = 0, \ldots, s - 1$.

Assume $\Lambda_j \cap \overline{\Pi}_i$ is an $r$-dimensional subspace of $\overline{\Pi}_i$ for some index $i$; since $\sigma$ fixes $\overline{\Pi}_i$ and permutes $\Lambda_0, \ldots, \Lambda_{m-1}$ transitively, we see that each one of $\Lambda_0 \cap \overline{\Pi}_i, \ldots, \Lambda_{m-1} \cap \overline{\Pi}_i$ is an $r$-dimensional subspace of $\overline{\Pi}_i$. The subspaces $\Lambda_0 \cap \overline{\Pi}_i, \ldots, \Lambda_{m-1} \cap \overline{\Pi}_i$ are pairwise skew and generate $\overline{\Pi}_i$ and so, since $\overline{\Pi}_i$ has dimension $m - 1$, we cannot have $r0$.

In order to show that the subspaces $\Lambda_0, \ldots, \Lambda_{m-1}$ cannot be simultaneously skew to $\overline{\Pi}_i$ it is thus sufficient to prove that $\overline{\Pi}_i$ contains a point represented by an eigenvector of $U^{(q^{n-1})/(q^{m-1})}$. Now the group $\hat{T}$ generated by this transformation induces a Singer cyclic group on $\overline{\Pi}_i$ and so we know that $GF(q^m)$ contains all eigenvalues of $U^{(q^{n-1})/(q^{m-1})}$, whence the assertion.

We conclude that each subspace $\Lambda_j \cap \overline{\Pi}_i$ consists of precisely one point $P_j$ for $j = 0, 1, \ldots, m - 1$. The points $P_0, \ldots, P_{m-1}$ are conjugate over $GF(q)$ with respect to $GF(q^m)$ and span $\overline{\Pi}_i$ in $PG(n - 1, q^m)$. \[\square\]