CHARACTERIZATION OF THREE TYPES OF PLANAR SPACES WITH INVISIBLE PLANES

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In this paper we characterize a class of finite planar spaces whose planes have two sizes.

1 GENERALITIES ON PLANAR SPACES

A linear space is a pair $(S, \mathcal{L})$, where $S$ is a set whose elements are called points and $\mathcal{L}$ is a family of subsets of $S$ called lines, such that through any pair of distinct points there is exactly one line, there are at least two lines and every line has at least two points. If $(S, \mathcal{L})$ is a finite linear space, then the number $[p]$ of lines through a point $p$ is called the degree of $p$ and the cardinality $|\ell|$ of a line $\ell$ is called the length of $\ell$. Put $n+1 = \max\{[p], \ p \in S\}$. Then $n$ is called the order of the space. For every point-line pair $(p, \ell)$ with $p \notin \ell$, we denote $\pi(p, \ell)$ the number $[p] - |\ell|$ of lines through $p$ parallel to $\ell$; i.e. the lines through $p$ missing $\ell$. If $H$ is a finite set of non-negative integers, the linear space $(S, \mathcal{L})$ is called a H-semiaffine plane if we have $\pi(p, \ell) \in H$ for any point-line pair $(p, \ell)$ with $p \notin \ell$. A subset $T$ of $S$ is called a subspace if it contains the line through any pair of distinct points of $T$.

A planar space is a triple $(S, \mathcal{L}, \mathcal{P})$, where $(S, \mathcal{L})$ is a linear space and $\mathcal{P}$ is a family of subspaces of $(S, \mathcal{L})$, called planes, such that

- through any three non-collinear points there is exactly one plane;
- every plane contains at least three non-collinear points;
- there exist at least two planes.

Let $(S, \mathcal{L}, \mathcal{P})$ be a finite planar space. For every plane $\pi$ of $\mathcal{P}$ we will denote by $\mathcal{L}_\pi$ the set consisting of the lines of $\mathcal{L}$ contained in $\pi$ and by $n(\pi)$ the order of the linear space $(\pi, \mathcal{L}_\pi)$.

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The integer \( n = \max n(\pi) \) is called the *order* of the planar space. We will denote by \( v \) the number of points, by \( b \) the number of lines and by \( c \) the number of planes.

A planar space \((S, L, P)\) is said to be *embeddable* into a projective space \( P \) if there is an injection of \( S \) into \( P \) which preserves, together its inverse map, collinearities and complanarities. Concerning with the problem of embeddability of a finite planar space into a three dimensional projective space, the following two properties play a crucial role.

(P1) Two distinct planes intersect either in the empty set or in a line.

(P2) In every plane \( \pi \) the points of the linear space \((\pi, L_\pi)\) have all the same degree \( n+1, n \geq 2 \).

The following theorem holds

**THEOREM** [5]. Let \((S, L, P)\) be a finite planar space of order \( n \) satisfying (P1) and (P2). If \(|S| \geq n^3 - n^2 + n + 2\), then \((S, L, P)\) is embeddable into the three dimensional projective space of order \( n \).

Let us now analyze some basic properties of a finite planar space satisfying (P1) and (P2). If \( p \) is a point of the planar space \((S, L, P)\), then \((\pi_p, L_p)\) will be the linear space having as "points" the lines through \( p \) and as "lines" the planes through \( p \) and such that the incidence relation is the inclusion. In view of (P1) and (P2), this gives a projective plane of order \( n \). It follows that if \( \pi \) is a plane of the planar space \((S, L, P)\), then the linear space \((\pi, L_\pi)\) has order \( n \) and it is embeddable into a projective plane of order \( n \). To see this take a point \( p \) not in \( \pi \) and consider the map that maps each point \( x \) of \( \pi \) to the line \( px \), "point" of \((\pi_p, L_p)\). This map is an embedding from the linear space \((\pi, L_\pi)\) into the projective plane \((\pi_p, L_p)\).

It is easy to see that (P1) and (P2) are equivalent to (P1) and (P2)

(P2)' In every plane \( \pi \) all the points of \((\pi, L_\pi)\) have degree at least three.

Let \((S, L, P)\) be a space satisfying (P1) and (P2). Since \((\pi_p, L_p)\) is a projective plane of order \( n \) for any point \( p \), the following properties hold:

- (a) through any point there are \( n^2 + n + 1 \) lines and \( n^2 + n + 1 \) planes;
- (b) every plane has at most \( n^2 + n + 1 \) lines;
- (c) through every line there are \( n + 1 \) planes;
- (d) every line has at most \( n + 1 \) points;
- (e) every plane contains at most \( n^2 + n + 1 \) points;
- (f) \( v \leq n^3 + n^2 + n + 1 \).

To classify all finite planar spaces whose planes have only two cardinalities seems to be a very difficult problem. The authors tackle this problem under the more restrictive hypothesis that small planes (i.e. the invisible planes) are "quite few". To pursue this aim they suppose that through any line of the space there are at most two small planes. The definitions in the next section will make this idea precise.