A COMBINATORIAL CHARACTERIZATION OF THE KLEIN QUADRIC

In memoriam Giuseppe Tallini

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We give a combinatorial characterization of the Klein quadric in terms of its incidence structure of points and lines. As an application, we obtain a combinatorial proof of a result of Havlicek.

1. INTRODUCTION

The celebrated Segre’s Theorem on irreducible conics of a Galois plane of odd order is the seminal result for the characterization problem of the incidence geometry of suitable projective algebraic varieties.

Some results for elliptic quadrics in PG(n,q) have been obtained by Barlotti [1] and Panella [6] and for general quadrics by Tallini [7] and [8]. In this context the notion of Tallini set (name suggested by Lefèvre [5]) naturally arose.

Let $\mathbb{P}=\text{PG}(n,K)$ be a projective space on a skew-field $K$. A Tallini set of $\mathbb{P}$ is a proper set of points spanning $\mathbb{P}$ and containing any line with at least three points in common with it. A Tallini set $T$ is called ruled if it contains a line. A ruled Tallini set will be denoted by $(T,L)$, where $L$ is the set of lines of $\mathbb{P}$ contained in $T$.

Examples of Tallini sets are quadrics and intersection of quadrics. Observe that the theory of non ruled Tallini sets coincide with the theory of caps and, trivially, each point set of $\text{PG}(n, 2)$ is a Tallini set. So, we may suppose the Tallini set ruled and the skew-field $K$ different from $\mathbb{Z}_2$.

Two points $x$ and $y$ of a Tallini set $(T,L)$ are called collinear if the line $xvy$ of $\text{PG}(n, K)$, joining $x$ and $y$, belongs to $L$. More generally, two subsets $X$ and $Y$ of $T$ are said to be collinear if each point of one of them is collinear with all the points of the other one. The symbol $x\sim_y (X \sim Y)$ means that the two points $x$ and $y$ (subsets $X$ and $Y$) are collinear.

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For any subset $X$ of $T$, $X^\perp$ denotes the set of points of $T$ collinear with all the points of a subset $X$. A point $x$ of $T$ is called \emph{singular} if it is collinear with all the other points of $T$, i.e. $x^\perp = T$. A singular Tallini set is one with singular points. It is easy to see that the set $W$ of singular points of $T$ is a subspace of $\text{PG}(n, K)$ and $T$ is a cone projecting from $W$ a non singular Tallini set of a subspace $W'$, complementary to $W$ [9].

A ruled Tallini set is connected if the collinearity graph of $T$ is connected, i.e. for every two points $x, y$ of $T$ there exists a chain $x_0 = x, x_1, \ldots, x_h = y$ with $x_i \neq x_{i+1}$.

Tallini sets seem to be a good approach for the combinatorial characterization of algebraic varieties which are intersection of quadrics even in non finite case. Actually, the results of Buekenhout and Lefèvre [2, 3] on quadrics, semiquadratic and quadratic sets of a desarguesian projective space $\mathbb{P}$ can be included in this program. Recall that:

- a \emph{semi-quadratic set} is a set of points of a projective space $\mathbb{P}$ such that the union of tangent lines at any point is a hyperplane of $\mathbb{P}$ or $\mathbb{P}$ itself;
- a \emph{quadratic set} is a semi-quadratic Tallini set;
- a \emph{Shult space} is a ruled point set $\mathcal{A}$ spanning $\mathbb{P}$ and such that for every line $L$ and any point $x$ of $\mathcal{A}$, either $L$ is contained in $x^\perp$ or $|L \cap x^\perp| = 1$.

**THEOREM I** (Buekenhout and Lefèvre [3]). Every non singular Shult space of a finite dimensional projective space is a semi-quadratic set.

**THEOREM II** (Buekenhout [2]). If $\mathcal{B}$ is a non singular ruled quadratic set of a finite dimensional projective space $\text{PG}(n, K)$ over a skew-field $K$, then $K$ is a field and $\mathcal{B}$ is a quadric.

In this paper we obtain a characterization of the Klein quadric as a connected non singular Tallini set $(T, L)$ of a desarguesian projective space of finite dimension, satisfying the following condition

\[(Q) \quad \text{for every pair } L, M \text{ of non collinear skew lines of } T, L^\perp \cap M^\perp \text{ either contains only two points or intersects both } L \text{ and } M.\]

Precisely, we prove the following result.

**THEOREM III.** Let $(T, L)$ be a connected, non singular Tallini set of a projective space $\text{PG}(n, K)$ on a skew-field $K$ different from $\mathbb{Z}_2$, satisfying condition $(Q)$. Then $K$ is a field, $n = 5$ and $T$ is the hyperbolic quadric $Q_5^+(K)$ of $\text{PG}(5, K)$.

Let now $\mathbb{P}$ be a projective space. The Grassmann geometry $\text{Gr}(h, \mathbb{P})$ is the incidence geometry whose points are the $h$-subspaces of $\mathbb{P}$ and whose lines are the \emph{pencils} of $h$-subspaces (a pencil being the set of $h$-subspaces contained in a $(h+1)$-subspace and passing through a $(h-1)$-subspace). If $\mathbb{P}$ is a projective space $\text{PG}(m, F)$ of finite dimension $m$ over a skew-field $F$, $\text{Gr}(h, \mathbb{P})$ is also