ON THE VARIATIONAL EQUATIONS ASSOCIATED WITH A LAGRANGIAN

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Abstract. In a previous publication, Broucke [1] has studied the symplectic properties of the variational equations of a Lagrangian of a very particular form, with constant coefficients. In this article, we generalize his results to the case of an arbitrary Lagrangian. We show that the characteristic exponents of a periodic solution can be computed in Lagrangian formulation as well as in the more usual Hamiltonian formulation.

1. Introduction: The Floquet Theorem

Let us assume a system of differential equations $\dot{x} = f(x)$ with a known periodic solution $x(t)$. The variational equations $\dot{\xi} = A(t)\xi$ associated with this periodic solution have then a general solution

$$\xi = R\xi_0; \quad R(0) = I; \quad \dot{R} = AR.$$ 

Here $R$ is called the resolvant or fundamental matrix. We know from the Floquet theory that

$$R = \Pi e^{Bi}; \quad \Pi(0) = I; \quad R(T) = e^{BT},$$

where $\Pi$ is a periodic matrix and $B$ a constant matrix [2, 3]. Note that if the periodic solution $x(t)$ is replaced by an equilibrium point, $\Pi$ is then the identity matrix and $B$ is equal to $A$. In the general case, a relation between $B$ and $A$ can also be obtained. Let us differentiate the above formula for $R$:

$$\dot{R} = \dot{\Pi}e^{Bi} + \Pi Be^{Bi} = A\Pi e^{Bi},$$

which gives

$$A\Pi = \dot{\Pi} + \Pi B; \quad A = \dot{\Pi}\Pi^{-1} + \Pi B\Pi^{-1},$$

with the initial value $A(0) = \dot{\Pi}(0) + B$.

It is well known that the stability of periodic solutions is usually derived from the eigenvalues of the monodromy matrix $R(T) = e^{TB}$. These eigenvalues, $\lambda$, sometimes called characteristic multipliers, are also related to the eigenvalues $\alpha$ of the matrix $B$ by the familiar relationship $\lambda = e^{\alpha T}$. In other words, the linear stability of a periodic solution depends solely on the constant matrix $B$. The object of the present note is to show that this stability calculation can be performed in Lagrangian formalism as well as in Hamiltonian formalism, as was shown by Broucke for a particular case [1].
2. Variational Equations of Hamiltonian Systems

In the case where the dynamical system has a Hamiltonian $H(q, p)$ and the canonical equations of motion $\dot{x} = JH_x$, with $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, we have the variational equations

$$\ddot{\xi} = JH_{xx}\dot{\xi} = A(t)\dot{\xi},$$

with $\dot{\xi}^T = [\delta q, \delta p]$. It is easy to see that the canonical variational equations have a remarkable first integral (Poisson bracket) $[4, 5, 6]: \omega = \dot{\xi}^T J \dot{\xi} = \text{Const}$. Indeed, by differentiating this quantity, we see that

$$\dot{\omega} = \dot{\xi}^T H_{xx}\dot{\xi} - \dot{\xi}^T H_{xx}\dot{\xi} = 0.$$

Several interesting properties can be derived from the existence of this integral. Replacing for instance, $\dot{\xi}$ by $R\dot{\xi}$, gives an important property of $R$:

$$\dot{\xi}^T R^T J R \dot{\xi} = \dot{\xi}^T J \dot{\xi},$$

$$R^T J R = J.$$

This is the symplectic property of the fundamental matrix $R$ [7]. As a consequence of this, we know that the eigenvalues of $R$ come in reciprocal pairs.

3. Variational Equations of a Lagrangian System

Here we have a dynamical system defined by an arbitrary Lagrangian $L(q, q, t)$ [1]. We also assume that we have a periodic solution, with period $T$, and the corresponding variational equations

$$\ddot{\eta} = A_L(t)\dot{\eta}.$$

Here $\eta$ represents the column vector with components $\delta q$ and $\delta \dot{q}$. Also, the subscript $L$ indicates that the quantity $A_L$ refers to a Lagrangian formulation. The solution of the variational equations satisfy relations similar to those that were found in the two preceding sections. We write them here with a subscript $L$:

$$\eta = R_L\eta_0; \quad R_L(0) = I; \quad \dot{R}_L = A_L R;$$

$$R_L = \Pi_L e^{B_L T}; \quad \Pi_L(0) = I; \quad R_L(T) = e^{B_L T}.$$

Here again, $\Pi_L$ is periodic, with period $T$, and $B_L$ is constant. We also have the relation

$$A_L = \dot{\Pi}_L \Pi_L^{-1} + \Pi_L B_L \Pi_L^{-1}.$$

However, in order to discover more properties of the Lagrangian matrices, we first establish the connection with the corresponding Hamiltonian matrices. We know that we have the definition of the momenta: $p = \partial L/\partial \dot{q} = L_q$. As for the variation, we have