AN INTERMEDIATE MATCHING TECHNIQUE FOR SOLVING
TWO POINT BOUNDARY VALUE PROBLEMS USING THE
PERTURBATION METHOD

W. E. WILLIAMSON*

The University of Texas, Austin, Texas

(Received 15 March, 1971)

Abstract. The perturbation method, a numerical method for solving two point boundary value
problems (TPBVP), is modified to attempt to improve inherent instability and sensitivity problems
associated with the method. The desired solution to the TPBVP is divided into two time intervals. The
differential equations required to define a solution to the two point boundary value problem are
integrated independently over these shorter segments rather than consecutively over the entire
trajectory. The independent integration of the differential equations over approximately half of the
trajectory instead of the entire trajectory substantially decreases sensitivity and stability properties
associated with the numerical integration. The equations for both time segments can be integrated
simultaneously. By this procedure, a system of twice the dimension of the original problem is inte-
grated for a period of time equal to half of the time interval for the original problem. To show the
effectiveness of the method, two impulse trajectories which minimize the total velocity increment
required to transfer a spacecraft from an Earth orbit into a lunar orbit are calculated.

1. Introduction

The application of linear perturbation equations to the problem of solving nonlinear
two point boundary value problems is extensive in the literature. See for example
Goodman and Lance (1956), Breakwell et al. (1963), Tapley and Lewallen (1967) and
Williamson (1970). The instability of the perturbation equations and the sensitivity
of the system of nonlinear equations to small changes in guessed boundary conditions
are the major problems associated with using this method to solve nonlinear TPBVP.
A method of implementing the perturbation algorithm which minimizes these diffi-
culties will be discussed in this paper.

The perturbation method requires that a system of linear differential equations be
integrated along the entire trajectory. Exponential type solutions are obtained for the
linear system of equations. If the eigenvalues of the system differ substantially in
magnitude and if the system is integrated over a sufficiently long time interval then
information about the desired solution may be lost. That is, information about the
small exponential type terms will be lost in the integration error of the larger expo-
nential type terms.

Integration characteristics may be improved by increasing the machine work length,
changing the eigenvalues of the system, or decreasing the integration time interval.
For some problems, double precision arithmetic should substantially improve
numerical stability problems. Choosing the proper variables or using a smoothing

* W. E. Williamson, Assistant Professor, Department of Aerospace Engineering and Engineering
Mechanics, University of Texas at Austin, Austin, Texas.
transformation similar to the one discussed in Williamson (1970) may improve stability problems.

As an alternate approach, decreasing the integration time interval should improve stability problems. This may be accomplished by dividing the trajectory into two segments. Unknown variables are guessed at both the initial and the final time. All of the necessary equations are integrated independently from \( t_0 \) and from \( t_f \) toward an intermediate time. At the intermediate time, the dependent variables and independent variable are required to be continuous. This approach is suggested by Fox (1962).

The approach allows both the nonlinear and the perturbation equations to be integrated independently over approximately half of the trajectory. Instability and sensitivity problems should thus be improved.

Another problem encountered using the standard perturbation method is that missing boundary conditions are guessed at only one point in time, usually \( t_0 \). Then the nonlinear differential equations are integrated to \( t_f \). Very little choice for the initial nominal trajectory near \( t_f \) is possible. By guessing unknown variables at \( t_0 \) and \( t_f \) the programmer has more flexibility in the choice of the initial starting nominal trajectory.

The method will be developed here for an even number of differential equations, which is characteristic of optimal control problems. The extension to systems with an odd number of equations is obvious.

2. Description of the Problem

The two point boundary value problem may be defined as follows: Determine the solution to

\[ \dot{z} = F(z, t) \]  

where \( z \) is a \( 2n \) vector of dependent variables, \( t \) is the independent variable, and \( F \) is a \( 2n \) vector of derivatives of \( z \). Equation (1) is required to satisfy the following boundary conditions

\[ h_0(z_0, t_0) = 0 \]  
\[ h_f(z_f, t_f) = 0 \]

where \( h_0 \) and \( h_f \) are \( n+1 \) vectors of initial and terminal conditions respectively. The standard perturbation method involves satisfying as many of the conditions from \( h_0 = 0 \) as possible. The choice of the starting point, either \( t_0 \) or \( t_f \) is irrelevant. Here it is chosen as \( t_0 \) for convenience. Then values are assumed for the unknown variables at \( t_0 \) and a value for \( t_f \) is guessed. A nominal trajectory is integrated from \( t_0 \) to \( t_f \). In general the nominal does not satisfy Equation (3) and possibly some of the equations in Equation (2).

The linear perturbation equations are

\[ \delta z = A(t) \delta z \]  

where \( A(t) \) is the \( 2n \times 2n \) matrix of partial derivatives of \( F \) with respect to \( z \), i.e. \( A(t) = \)