PERIODIC ORBITS AROUND AN OBLATE SPHEROID

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Abstract. The general properties of certain differential systems are used to prove the existence of periodic orbits for a particle around an oblate spheroid.

In a fixed frame, there are periodic orbits only for \( i = 0 \) and \( i \) near \( \pi/2 \). Furthermore, the generating orbits are circles.

In a rotating frame, there are three families of orbits: first a family of periodic orbits in the vicinity of the critical inclination; secondly a family of periodic orbits in the equatorial plane with \( 0 < e < 1 \); thirdly a family of periodic orbits for any value of the inclination if \( e = 0 \).

1. Introduction

The present paper aims to apply the general properties of nearly-integrable differential systems in order to demonstrate the existence of classes of periodic solutions for the motion of a particle which gravitates around an oblate spheroid.

The potential is limited to the \( J_2 \) term. The equations of the satellite's motion form a differential system which is close to an integrable system. The properties of periodic solutions for such a system are described by Roseau (1966), Haag (1948) and others. By applying these properties we find all the periodic solutions in the vicinity of the periodic solutions of the integrable systems. Among them are some solutions found by other authors with different methods, e.g. MacMillan (1920) and more recently Delmas (1978), Kammeyer (1976).

The main interest of the described method is probably the fact that the formalism is the same for all classes of orbits and a great part of calculations has to be done only once.

2. Equations and Method

The method we use was described in previous papers by Stellmacher (1976; 1977; 1979). We summarize the main points.

In a cartesian frame, the equations of the motion are given by the autonomous system

\[
\dot{x} = f(x) + J_2 g(x) .
\]

Equation (1) defines a sixth order system; \( x \), \( f(x) \) and \( g(x) \) are vectors with six components; \( f(x) \) and \( g(x) \) have successive continuous derivatives with respect to \( x \). \( J_2 \) is a small quantity which is approximatively \( 10^{-3} \) in the Earth's case. For \( J_2 = 0 \), system (1) is reduced to

\[
\dot{x} = f(x) .
\]
System (2) has a family of elliptical solutions depending on six parameters $\rho_i$, $i = 1 \ldots 6$; the elliptical elements for instance. Note that the period $T$ of these solutions, is a function of one parameter $\rho_i$, i.e., the semimajor axis of the ellipse.

The question is: Does (1) possess a (or several) periodic solution which tends to a (or several) solution of (2) for $J_2 \to 0$.

System (1) is an autonomous system, its periodic solution, if any, has a period $T'$ close to $T$; we put $T' = T/(1 + J_2 \delta)$, where $\delta$ is a continuous function of $J_2$ expanded in a power series $\delta = \delta_0 + J_2 \delta_1 + \cdots$. System (1) can now be written

$$\begin{align*}
\dot{x} &= (1 + J_2 \delta) f(x) + J_2 \tilde{g}(x, J_2) \\
\tilde{g}(x, J_2) &= g(x) - \delta f(x).
\end{align*}$$

We define as main system associated with (1a), the system

$$\dot{x} = (1 + J_2 \delta) f(x).$$

System (2a) has a family of periodic solutions, with period $T'$, depending on six parameters. Let $z(t, \rho_i)$ be this family of solutions; $\rho_i$ can be arbitrarily chosen.

The variational equation is

$$\dot{y} = (1 + J_2 \delta) Q(t) y.$$ 

$Q(t)$ is the matrix $6 \times 6$ of elements $(\partial f/\partial x_i)_{x=z}$. System (3) is a linear differential system with periodic coefficients. It has a set of independent solutions $\varphi^i = \partial z/\partial \rho_i$, $i = 1 \ldots 6$. Let $\Phi$ be the matrix of elements $\varphi^i$.

For $\rho_i \neq \rho_p$, $\varphi^i$ are periodic solutions with period $T'$. If $\rho_i = \rho_p$, $z(n(t'), \rho_i)$ is solution with period $T$ with regard to $t'$ of the system $dx/dt' = f(x)$; $t' = (1 + J_2 \delta) t$ and $n = 2\pi/T$ is a function of $\rho_i$; $z(n(t'-\gamma), \rho_i) = z(l, \rho_i)$ is also solution of this system.

$$\begin{align*}
\varphi^i &= \delta z + \frac{\partial z}{\partial \rho_i} + \frac{\partial l}{\partial \rho_i} + \frac{\partial n}{\partial \rho_i} \\
\varphi^i &= \frac{\partial z}{\partial \rho_i} - \frac{l}{n^2} \frac{\partial n}{\partial \rho_i} \\
\varphi^i &= \bar{\varphi} + K t \varphi^{i-1},
\end{align*}$$

with $K = (-1/n^2)(\partial n/\partial \rho_i)$ and $\rho^{i-1} = \gamma$. $\delta$ means the explicit derivation and $\varphi^i$ is a solution of $dy/dt' = Q(t')y$. With variable $t$, $\varphi^i$ is solution of (3). $\bar{\varphi}$ is a periodic function with period $T'$, $l = n(t'-\gamma) = n(1 + J_2 \delta) t - n \gamma = n't - n \gamma$.

**Remark.** The parameters $\rho_i$ can be ordered in any way. If the semi major axis carries the index $j$, we only admit that $\gamma$ carries the index $j - 1$.

The adjoint system to (3)

$$\dot{y} = -(1 + J_2 \delta) Q(t)^* y$$

has a set of six solutions $\psi^j$; among them five solutions are periodic with period $T'$, one solution is $\psi^{j-1} = \bar{\psi} - Kl \psi^j$. $\bar{\psi}$ is a periodic function with period $T'$. Let $\Psi$ be the matrix of elements $\psi^j$; $\psi^j$ is the column vector associated with the column vector $\varphi^i$ in the transformation $\Psi = (\Phi^{-1})^*$. 