Explicit formulas of minimum-variance linear unbiased estimators of the unknown mean are derived for homogeneous random fields with correlation functions

\[ B(r_1, r_2) = A e^{-\alpha_1|t_1| - \alpha_2|t_2|}; \]

observed on the set

\[ k_{r_1, r_2} = \{(t_1, t_2) : |t_1| \leq T_1, t_2 = 0\} \cup \{(t_1, t_2) : |t_1| \leq T_1, t_1 = 0\} \]

(a "cross").

Assume that the random field

\[ \xi(t_1, t_2) = a + \eta(t_1, t_2), \]

where \( a \) is an unknown parameter, \( \eta(t_1, t_2) \) is a homogeneous random field with a known correlation function and zero mean, is observed on some set \( E \) in the plane. It is required to estimate the parameter \( a \), i.e., the mean of the random field \( \xi(t_1, t_2) \).

It is easy to show (for comparison, see A. M. Yaglom’s supplement to [1]) that the minimum-variance linear unbiased estimator \( \hat{a}_E \) of the parameter \( a \) is determined for some \( c \) from the equation

\[ \hat{a}_E(t_1, t_2) = c \eta(t_1, t_2) \quad \forall (t_1, t_2) \in E. \]

The constant \( c \) in Eq. (2) is the variance of the estimator \( \hat{a}_E \) in some special regions \( E \).

We consider the case when \( E \) is a "cross,

\[ K_{r_1, r_2} = \{(t_1, t_2) : |t_1| \leq T_1, t_2 = 0\} \cup \{(t_1, t_2) : |t_1| \leq T_1, t_1 = 0\}. \]

The problem of estimation of the spectral density of a random field observed on a "cross" was considered by Konyaev [2].

We will calculate the optimal estimator \( \hat{a}_{T_1, T_2} \) in two particular cases:

a) the correlation function of \( \eta(t_1, t_2) \) is

\[ B(r_1, r_2) = A e^{-\alpha_1|t_1| - \alpha_2|t_2|}; \]

b) the correlation function of \( \eta(t_1, t_2) \) is

\[ B(r_1, r_2) = \max \left\{ 0,1 - \frac{|r_1|}{d} \right\} \max \left\{ 0,1 - \frac{|r_2|}{d} \right\}. \]

**THEOREM 1.** Assume that the random field \( \xi(t_1, t_2) \) with the correlation function (3) is observed on \( K_{T_1, T_2} \). Then the linear unbiased estimator \( \hat{a}_{T_1, T_2} \) of the unknown mean \( a \) is given by the equality

\[ \hat{a}_{T_1, T_2} = \frac{1}{2} \left[ \xi(T_1, 0) + \xi(0, T_2) + 2 \xi(0, 0) \right] + \xi(T_1, 0) + \xi(0, -T_2) + \alpha_1 \int_{-T_1}^{T_1} \xi(t_1, 0) dt_1 + \alpha_2 \int_{-T_2}^{T_2} \xi(0, t_2) dt_2. \]
The variance of the estimator $\tilde{a}_{T_1, T_2}$ is

$$D\tilde{a}_{T_1, T_2} = \frac{A}{1 + \alpha_1 T_1 + \alpha_2 T_2}.$$  \tag{6}

**Proof.** It is easy to verify by direct computation that the estimator $\tilde{a}_{T_1, T_2}$ satisfies Eq. (2) with the constant $c$ given by (6). By uniqueness of the estimator, this proves the theorem.

Let us now present some "heuristic" considerations which have led to the estimator (5). We seek the estimator in the form

$$\tilde{a}_{T_1, T_2} = \int_{-T_1}^{T_1} c_1(t_1) \delta(t_1, 0) \, dt_1 + \int_{-T_2}^{T_2} c_2(t_2) \delta(0, t_2) \, dt_2,$$  \tag{7}

where $c_1(t_1)$ and $c_2(t_2)$ are unknown weights. By the unbiasedness condition, the weights should satisfy the condition

$$\int_{-T_1}^{T_1} c_1(t_1) \, dt_1 + \int_{-T_2}^{T_2} c_2(t_2) \, dt_2 = 1.$$  \tag{8}

Substitute (7) in the main equation (2). First consider this equation for points of the form $(s_1, 0)$:

$$M_{\tilde{a}_{T_1, T_2}} \eta(s_1, 0) = \int_{-T_1}^{T_1} A e^{-\alpha t_1} c_1(t_1) \, dt_1 + \int_{-T_2}^{T_2} A e^{-\alpha s_2} c_2(t_2) \, dt_2 = c.$$  \tag{9}

Twice differentiating both sides of Eq. (9) with respect to $s_1$, we obtain for $-T_1 < s_1 < T_1$

$$\frac{\partial^2 M_{\tilde{a}_{T_1, T_2}} \eta(s_1, 0)}{\partial s_1^2} = \alpha^2 c - 2 \alpha A c_1(s_1) = 0.$$  

Hence,

$$c_1(s_1) = \frac{\alpha^2 c}{2A} \text{ for } s_1 \in (-T_1, T_1).$$

Now consider the main equation for the points $(0, s_2)$ ($-T_2 \leq s_2 \leq T_2$):

$$M_{\tilde{a}_{T_1, T_2}} \eta(0, s_2) = \int_{-T_2}^{T_2} A e^{-\alpha s_2} c_2(t_2) \, dt_2 + \int_{-T_1}^{T_1} A e^{-\alpha s_2} c_1(t_1) \, dt_1,$$  \tag{10}

Differentiating both sides of Eq. (10) with respect to $s_2$, we obtain

$$\frac{\partial^2 M_{\tilde{a}_{T_1, T_2}} \eta(0, s_2)}{\partial s_2^2} = \alpha^2 c - 2 \alpha A c_2(s_2) = 0.$$  

Hence,

$$c_2(s_2) = \frac{\alpha^2 c}{2A} \text{ for } s_2 \in (-T_2, T_2).$$

In order to allow for the "boundary effect," we seek $c_1(t_1)$ and $c_2(t_2)$ in the form

$$c_1(t_1) = \gamma_1 \delta(t_1 - T_1) + \gamma_2 \delta(t_1 + T_1) + \gamma_3 \delta(t_1) + \frac{\alpha^2 c}{2A},$$  \tag{11}

$$c_2(t_2) = \beta_1 \delta(t_2 - T_2) + \beta_2 \delta(t_2 + T_2) + \beta_3 \delta(t_2) + \frac{\alpha^2 c}{2A},$$  \tag{12}