Mailbox

On V. Fleischer's characterization of absolutely flat monoids

SYDNEY BULMAN-FLEMING\textsuperscript{1} and KENNETH McDOWELL\textsuperscript{2}

Abstract. In his paper *Completely flat monoids* (Učh. Zap. Tartu Un-ta 610 (1982), 38–52 (Russian)) V. Fleischer gives a characterization of the absolute flatness of a monoid $S$ in terms of certain one-sided ideals and one-sided congruences of $S$. In the present work an alternative, more direct proof of Fleischer's theorem is provided, and the result is used to show that the multiplicative monoid of any semisimple Artinian ring is absolutely flat.

\section*{Preliminaries}

If $S$ is a monoid $S$-Ens (respectively, Ens-$S$) will denote the class of all left (right) unital $S$-sets. For $A \in$ Ens-$S$ and $B \in S$-Ens the tensor product $A \otimes B$ is defined to be the set $(A \times B)/\tau$, where $\tau$ is the smallest equivalence relation on $A \times B$ containing all pairs $((as, b), (a, sb))$ for $a \in A$, $b \in B$, and $s \in S$. For $a \in A$ and $b \in B$, $a \otimes b$ represents the $\tau$-class of $(a, b)$. $A \in$ Ens-$S$ (respectively, $B \in S$-Ens) is called flat if the functor $A \otimes -$ (respectively, $- \otimes B$) preserves embeddings, and $S$ is called (left, right) absolutely flat if all of its (left, right) $S$-sets are flat. Absolutely flat monoids were first studied by M. Kilp [7]. The reader is referred to [1]–[4] for more recent work.

\section*{Fleischer's theorem}

In [5] V. Fleischer proves the following result.

THEOREM 1. A monoid $S$ is absolutely flat if and only if $S$ is regular and $S$ satisfies the conditions (L) and (R) below:

(L) for every $x, y \in S$ there exists $u \in xS \cap yS$ such that $(u, x) \in \theta_l(x, y)$.
(R) for every $x, y \in S$ there exists $v \in Sx \cap Sy$ such that $(v, x) \in \theta_r(x, y)$.

\textsuperscript{1} Research supported by Natural Sciences and Engineering Research Council grant A4494.
\textsuperscript{2} Research supported by Natural Sciences and Engineering Research Council grant A9241.
Presented by Boris M. Schein. Received June 19, 1987. Accepted for publication in final form October 23 1987.
Throughout this paper $\theta_L(x, y)$ and $\theta_R(x, y)$ denote the smallest left and right congruences, respectively, on $S$ which identify $x$ and $y$.

The following proof is more direct than that given in [5] and may be useful to readers to whom this reference is inaccessible.

Proof of Theorem 1. First assume $S$ is (left) absolutely flat. That $S$ is regular is contained in [7] (see also [1], Proposition 2.5). Suppose $x$ and $y$ are (without loss of generality distinct) elements of $S$. Because $S/\theta_L(x, y)$ is a flat left $S$-set, the canonical function $(xS \cup yS) \otimes S/\theta_L(x, y) \to S/\theta_L(x, y)$ which maps $z \otimes \bar{s}$ to $\bar{z}$ is injective. Since $\bar{x} = \bar{y}$ it follows that $x \otimes 1 = y \otimes 1$ in $(xS \cup yS) \otimes S/\theta_L(x, y)$.

By Lemma 1.1 of [1] there exist $x_1, \ldots, x_n \in xS \cup yS$, $s_1, \ldots, s_n \in S$, $t_1, \ldots, t_n \in S$ where $\{s_i, t_i\} = \{x, y\}$ for $i = 1, \ldots, n$ such that

$$
\begin{align*}
x &= x_1s_1 \\
x_1t_1 &= x_2s_2 \\
&\vdots \\
x_nt_n &= y.
\end{align*}
$$

Note that $(x, x_it_i) \in \theta_L(x, y)$ for each $i$. If $x_1 \in yS (L)$ is established by letting $u = x$. If $x_n \in xS$ one may take $u = y$. Otherwise there exists some $i$ for which $x_it_i \in xS \cap yS$ and this element is a suitable choice for $u$. ($R$) follows similarly from right absolute flatness.

Now assume $S$ is a regular monoid which satisfies conditions $(L)$ and $(R)$ and suppose $B$ is any left $S$-set. By symmetry it suffices to prove $B$ is flat. To this end, consider any right $S$-set $A$ and elements $a, \tilde{a} \in A$ and $b, \tilde{b} \in B$ such that $a \otimes b = \tilde{a} \otimes \tilde{b}$ in $A \otimes B$. By Lemma 1.2 of [1] there exist $a_1, \ldots, a_n \in A$, $b_2, \ldots, b_n \in B$, $s_1, \ldots, s_n \in S$, and $t_1, \ldots, t_n \in S$ such that

$$
\begin{align*}
a &= a_1s_1 \\
a_1t_1 &= a_2s_2 & s_1b &= t_1b_2 \\
a_2t_2 &= a_3s_3 & s_2b_2 &= t_2b_3 \\
&\vdots \\
a_nt_n &= \tilde{a} & s_nb_n &= t_nb.
\end{align*}
$$

(Such a system of equalities is called a scheme over $A$ and $B$ of length $n$ joining $(a, b)$ to $(\tilde{a}, \tilde{b})$.) By Lemma 2.2 of [1] we must show there exists a “replacement” scheme (of possibly different length) over $aS \cup \tilde{a}S$ and $B$ joining $(a, b)$ to $(\tilde{a}, \tilde{b})$.

For each $s \in S$ let $s'$ denote any inverse of $s$. The following lemma applies to elements from the scheme $(\Sigma)$ above.