IN Variance IN Von Zeipel Method

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Abstract. In this paper the relation between the Von Zeipel generating functions, in two canonical systems, is considered. Explicit recursive formulae between them are given. Also a series of invariants of the generating functions are obtained, so that the equations that appear in the Von Zeipel method can be written in terms of Poisson brackets.

1. Introduction

Among the perturbation methods employed in Celestial Mechanics, the Von Zeipel (1916–17) method plays a very important role. As compared with others based on the Lie derivative (Hori, 1966; Deprit, 1969), Von Zeipel method is conceptually more simple, nevertheless it presents some disadvantages with respect to those. Firstly, as the generating function depends on mixed variables, the transformation equations are written in implicit form; on the other hand, the partial differential equations, that determine the generating function in successive orders, have not invariant form and therefore the solution in the Von Zeipel method depends on the variables employed.

To avoid this difficulty, in this work we assume the knowledge of a Von Zeipel generating function and we want to find the Von Zeipel generating function in another canonical system of variables. Clearly this question is very important, because if a problem is integrated in one system of variables, it suffices to transform the generating function in order to solve it in another canonical system, and this may be applied to questions of convergence, elimination of small divisors, etc.

Finally, solving this problem we get a sequence of invariant functions that are the same as those which appear in successive orders of the Lie generator for the same transformation. Furthermore, it is clear that equations in the Von Zeipel method may be written advantageously in terms of the invariant functions because they appear in form of Poisson brackets which mean invariant form.

2. Transformation of a Generating Function

Consider the n-dimensional variables $X=(X_1, \ldots, X_n)$, $x=(x_1, \ldots, x_n)$, $X'=(X'_1, \ldots, X'_n)$, $x'=(x'_1, \ldots, x'_n)$ and let $S(X', x; \varepsilon)$ be a scalar function of $C(2)$ class in a certain $(2n+1)$-dimensional domain, with $\det(S_{x_i x_j}) \neq 0$, generating a completely canonical transformation $s: (X, x) \rightarrow (X', x')$, defined by

\[
X = S_x(X', x; \varepsilon)
\]

\[
x' = S_x(X', x; \varepsilon).
\]
With the same assumptions as in previous case, given a function $G(Y, x)$, the equations

$$
X = G_x(Y, x) \tag{2.2}
$$
$$
y = G_y(Y, x)
$$

define another completely canonical transformation $g: (X, x) \rightarrow (Y, y)$. Firstly, we shall try to obtain the generating function that permits us pass from $(Y, y)$ to $(X', x')$. With this purpose we shall prove the following proposition:

**PROPOSITION 1.** *The function*

$$
T(y, X'; e) = S - G + Y \cdot y \tag{2.3}
$$
*generates the transformation $t = s \cdot g^{-1}: (Y, y) \rightarrow (X', x')$. That is, the implicit equations of $t$ are*

$$
Y = T_y(y, X'; e) \tag{2.4}
$$
$$
x' = T_x(y, X'; e).
$$

In fact, the differential of Equation (2.3), using (2.1) and (2.2), becomes

$$
dT = dS - dG + d(Y \cdot y) = X \cdot dx + x' \cdot dX' - X' \cdot dx + y \cdot dY + Y \cdot dy + y \cdot dy = x' \cdot dX' + Y \cdot dy
$$

which owing to (2.4) is a total differential, and then $T$ is the generating function of $t$.

Let $g'$ be a transformation from $(x', X')$ to $(y', Y')$ which is formally the same as $g$. This means that this transformation can be obtained by putting primes to all variables in transformation $g$. It is clear that $g'$ is a completely canonical transformation given by the equations

$$
X' = G_x(Y', x') \tag{2.5}
$$
$$
y' = G_y(Y', x').
$$

Thus we have the following diagram

$$
\begin{array}{ccc}
(X, x) & \xrightarrow{g} & (Y, y) \\
\text{s} & \text{t} & \text{s}^* \\
(X', x') & \xrightarrow{g'} & (Y', y')
\end{array}
$$

where the meaning of $t$ is clear.

Finally, the transformation $s: (Y, y) \rightarrow (Y', y')$ is also completely canonical because it is a composition of transformations with the same properties. Its generating function is obtained in the following proposition: