THE CONTINUATION OF PERIODIC ORBITS FROM THE RESTRICTED TO THE GENERAL THREE-BODY PROBLEM

JOHN D. HADJIDEMETRIOU
University of Thessaloniki, Thessaloniki, Greece

(Received 21 December, 1973)

Abstract. It is proved that a symmetric periodic orbit of the circular planar restricted three-body problem can be continued analytically, when the mass of the third body is small but not negligible, to a periodic motion of the general three-body problem in a rotating frame of reference whose origin coincides with the center of mass of the two bodies with large masses and its x axis always contains these bodies. The two bodies with the large masses describe periodic motion on the x axis of the rotating frame while the third body, with the small mass, describes a symmetric periodic orbit in this frame. The motion of the two bodies lying on the x axis is always stable, whereas the periodic orbit of the third body in the rotating frame is stable or unstable depending on whether or not the nonzero characteristic exponents of the original periodic orbit of the restricted problem are of the stable or unstable type, respectively. It is also shown that for a fixed value of the small mass of the third body, a family of symmetric periodic orbits of the restricted problem can be continued analytically to a family of periodic motions of the general three-body problem.

1. Introduction

Several attempts have been made recently to obtain periodic solutions of the general planar three-body problem (e.g. Standish, 1969; Szebehely, 1969, 1971; Szebehely and Feagin, 1973). This problem is much more complicated than the restricted problem and most of the periodic orbits found are isolated for a fixed ratio of the masses, and correspond to the case where all three masses are finite and comparable with each other. In the present paper we consider the case where the third mass is finite but small compared to the masses of the other two bodies, so that we can ignore second and higher order terms in the small mass. We start with a periodic orbit of the planar restricted three-body problem and study the continuation of this orbit when the third mass is finite but small. As will be shown in the following, a symmetric periodic orbit of the restricted problem is in general continued as a symmetric periodic motion of the three bodies in the general problem.

We find it more convenient to use the Lagrangian equations of motion instead of the canonical equations, because in the latter case the small mass appears, formally, in the denominators in the expressions for the momenta (Whittaker, 1960), and this makes the separation of the small terms more difficult.

2. The Lagrangian function in a Rotating Frame

We consider three bodies $P_1, P_2, P_3$ with masses $m_1, m_2, m_3$ respectively, moving in the plane under their mutual gravitational attraction. We consider the system isolated
and, without loss of generality, we take the center of mass $G$ of the system to be at rest with respect to an inertial reference system. Then, if $\mathbf{R}_i$ are the position vectors of $P_i$ ($i=1, 2, 3$) (Figure 1), respectively, with respect to the center of mass $G$, we have

\[
\text{Fig. 1. The three bodies in the plane.}
\]

for the kinetic energy of the system

\[
T = \frac{1}{2} m_1 \dot{\mathbf{R}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{R}}_2^2 + \frac{1}{2} m_3 \dot{\mathbf{R}}_3^2, \tag{1}
\]

where the dot denotes differentiation with respect to the time.

We now define the vectors $\mathbf{r}_i$ by the relations

\[
\mathbf{r}_1 = \mathbf{G}_1 P_1, \quad \mathbf{r}_2 = \mathbf{G}_2 P_2, \quad \mathbf{r}_3 = \mathbf{G}_3 P_3, \tag{2}
\]

where $G_i$ is the center of mass of the two bodies $P_i, P_3$. Then, using the relation $m_1 \mathbf{R}_1 + m_2 \mathbf{R}_2 + m_3 \mathbf{R}_3 = 0$, which expresses that $G$ is the center of mass of the whole system, we obtain the following relations between $\mathbf{R}_i$ and $\mathbf{r}_i$:

\[
\mathbf{R}_1 = \mathbf{r}_1 - \frac{m_2}{m} \mathbf{r}_2, \quad \mathbf{R}_2 = \frac{m_1 + m_3}{m} \mathbf{r}_2, \quad \mathbf{R}_3 = \mathbf{r}_3 - \frac{m_2}{m} \mathbf{r}_2, \tag{3}
\]

where

\[
m = m_1 + m_2 + m_3. \tag{4}
\]

Substituting now the relations (3) into (1) and using the relation $m_1 \mathbf{r}_1 + m_3 \mathbf{r}_3 = 0$, which holds because $G_1$ is the center of mass of the bodies $P_1, P_3$, we obtain for the kinetic energy of the whole system

\[
T = \frac{1}{2} (m_1 + m_3) [q \dot{t}_2^2 + \dot{q} t_3^2], \tag{5}
\]

where

\[
q = \frac{m_2}{m}, \quad \dot{q} = \frac{m_3}{m_1}. \tag{6}
\]

The potential energy of the system is

\[
V = -\frac{k^2 m_1 m_3}{r_{13}} - \frac{k^2 m_2 m_3}{r_{23}} - \frac{k^2 m_1 m_2}{r_{12}}, \tag{7}
\]