AN EXTENDED IDEAL RESONANCE PROBLEM

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Abstract. An Extended Resonance Problem is defined by the Hamiltonian,
\[ F = B(y) + 2\mu^2 A(y)\sin x + \lambda(y)^2 \quad \mu \ll 1, \quad \lambda = O(\mu). \]
It is noted here that the phase-plane trajectories exhibit a double libration, enclosing two centers, for the initial conditions of motion satisfying the inequality
\[ 1 - |\lambda| < |x| < 1 + |\lambda|, \]
where \( \lambda \) is the usual resonance parameter.

A first order solution for the case of double libration is constructed here by a generalization of the procedure previously used in solving the Ideal Resonance Problem with \( \lambda = 0 \). The solution furnishes a reference orbit for a Perturbed Ideal Problem if a double libration occurs as a result of perturbations.

1. Introduction

An important class of problems in celestial mechanics is reducible to the Perturbed Ideal Resonance Problem. The latter is a problem of one degree of freedom, with the Hamiltonian of the form
\[ F = B(y) + 2\mu^2 A(y)\sin 2x + f(x, y), \quad \mu \ll 1, \quad f = o(1), \]
where \( f \) is a Fourier series in the critical argument \( x \). A first-order solution of the unperturbed problem, with \( f \equiv 0 \), is embodied in a series of papers numbered II–VI in the References. Under the assumption of normality (Paper V), the phase-plane diagram of the Ideal Resonance Problem is topologically equivalent to that of a simple pendulum, and is illustrated by the argument of the perigee in the Main Problem of the Artificial Satellite Theory (Paper VII). If \( f \neq 0 \), and if \( f \) contains the sine of an odd multiple of \( x \), the topological picture exhibits the novel feature of a double libration, enclosing two libration centers. In order to provide a reference orbit for (1) that preserves its topological character when double libration occurs in the actual motion, we consider the special case,
\[ f = A(y)(2\lambda \sin x + \lambda^2), \]
in which \( f \) is limited to a single trigonometric term, and \( \lambda^2 \) is inserted in order to simplify the analysis. This leads to the formulation of an Extended Problem with the Hamiltonian,
\[ F = B(y) + 2\mu^2 A(y)[\sin x + \lambda(y)]^2 \quad \lambda = O(\mu). \]

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That double libration occurs in (3) under certain initial conditions of the motion is shown in Section 2. A first-order solution of (3), including the ignorable coordinates, is then constructed for the case of double libration in Sections 3–9. This is done by a generalization of the procedures previously used in Papers II and VI for solving the Ideal Resonance Problem with $\lambda = 0$.

Another form of an Extended Problem exhibiting double libration was studied by Giacaglia (1970). In celestial mechanics double libration is illustrated by the horse-shoe orbits of the Trojan asteroids.

2. Phase-Plane Diagram

Generalizing the procedure used in Section 2 of Paper II, we replace Equation (15b)-II by

$$A = \pm [1 - \alpha^{-2}(\sin x + \lambda)^2]^{1/2}.$$  \hspace{1cm} (4)

With this new $A$, the phase-plane trajectories are given to the first order in $\mu$, as in (18)-II, by

$$y = y' + b_1(A - 1),$$ \hspace{1cm} (5)

where $\alpha$, $\lambda$, and $b_1$ are now functions of the constant $y'$. In terms of the variables $\xi$ and $\eta$ defined by

$$\xi = x + \frac{\pi}{2} \text{sgn} \lambda,$$

$$\eta = (y - y' + b_1)\alpha/b_1 = \alpha A,$$ \hspace{1cm} (6)

(5) assumes the form

$$f(\xi, \eta) = \frac{1}{2} [(\cos \xi - |\lambda|)^2 + \eta^2 - \alpha^2],$$ \hspace{1cm} (7)

symmetric with respect to $\xi$ and $\eta$-axes. The reader will not confuse $f(\xi, \eta)$ with $f$ of (1) and (2). Furthermore, since only $|\lambda|$ occurs in subsequent analysis, hereafter we shall write $\lambda$ for $|\lambda|$, for the sake of brevity.

The singular points of the family of curves (7), with $\alpha^2$ as the family-parameter, are determined from the relations

$$f_\xi = f_\eta = 0.$$ \hspace{1cm} (8)

The results of the analysis are displayed in Table I.

| Table I |
|---|---|---|---|---|
| $\xi$ | $\eta$ | $x$ | $|\alpha|$ | $D$ | Remarks |
| $0$ | $0$ | $-\pi/2$ | $1-\lambda$ | $-1+\lambda<0$ | saddle-point (inner) |
| $\pm \pi$ | $0$ | $\pi/2$ | $1+\lambda$ | $-1-\lambda<0$ | saddle-point (outer) |
| $\pi/2-\lambda$ | $0$ | $-\lambda$ | $0$ | $1-\lambda^2>0$ | center |
| $-\pi/2+\lambda$ | $0$ | $-\pi+\lambda$ | $0$ | $1-\lambda^2>0$ | center |