CLASSIFICATION OF LINEAR SPACES WITHOUT THREE MUTUALLY PARALLEL LINES

Dedicated to Giuseppe Tallini on the occasion of his 60th birthday

Klaus Metsch

We classify all finite linear spaces without three mutually parallel lines. Apart from two exceptions, such a space is necessarily a generalized projective plane, a simple extension of a generalized projective plane, or a complete inflated affine plane with a generalized projective plane at infinity.

1. INTRODUCTION

A (non-trivial) finite linear space is a finite set of points together with a family of subsets called lines such that any two distinct points p and q are in a unique line pq, every line contains at least two points, and there are at least two lines. The degree $r_p$ of a point p is the number of lines through p, and dually the degree $k_L$ of a line L is the number of points on L. Two lines are called parallel if they are disjoint. We denote the number of points by v, and the number of lines by b.

We are interested in linear spaces without three mutually parallel lines. The study of such spaces was motivated by results of A. Beutelspacher and A. Delandtsheer. They considered linear spaces satisfying a condition $D_j$, where j can be 0, 1, or 2; namely, there exists a non-negative integer d such that for any pair of parallel lines $L_1, L_2$ and point $p \notin L_1 \cup L_2$ one has $D_0$: exactly d lines through p intersect neither $L_1$ nor $L_2$;
$D_1$: exactly d lines through p intersect $L_1$ and not $L_2$;
$D_2$: exactly d lines through p intersect both $L_1$ and $L_2$.
The authors completely classified finite linear spaces satisfying $D_1$ [4] or $D_2$ [2]; $D_0$ however was settled only for $d>0$ [3]. For $d=0$ condition $D_0$ means that there does not exist three mutually parallel lines. In this paper, we shall determine all such spaces. To state our result, we need some more terminology.

For integers $r, k_1, ..., k_r \geq 2$, the $(k_1, ..., k_r)$-cross is the linear space with a point of degree $r$ contained in lines $L_j$ of degree $k_j$, $j=1, ..., r$, while every line not containing this point has degree 2. A $(2, ..., 2)$-cross is also called a complete graph, and a $(2, k)$-cross is also called a near-pencil.

A generalized projective plane is a projective plane or a near-pencil.

Inflated affine planes have been defined in [5]. Such a space $L$ consists of an affine plane $A$ together with a subset of its infinite points on which is imposed the structure of a linear space $D$. We also say that $L$ is an affine plane with $D$ at infinity. $L$ is called complete if the point set of $D$ is the set of all infinite points of $A$.

Suppose $L$ is a linear space. Then we can define a new linear space obtained from $L$ by adjoining a new point $\infty$ to $L$ and connecting it to every point of $L$ by a new line of degree 2. Similarly, we may add a new point $\infty$ to one line of $L$ and connect it to every point outside this line by a new line of degree 2. In both cases, we call the resulting linear space a simple extension of $L$.

Now we can state our result.

THEOREM. Every finite linear space without three mutually parallel lines is one of the following structures.

1. A generalized projective plane.
3. A complete inflated affine plane with a generalized projective plane at infinity.
4. The affine plane of order 3 with the affine plane of order 2 at infinity, or the complete graph on 5 points.