OVOIDS IN FINITE UNIFORM LINEAR SPACES

Dedicated to Professor Tallini on the occasion of his 60th birthday

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The concept of an ovoid is extended to linear spaces, and it is shown that a finite uniform linear space with an ovoid is either a trivial structure on a finite set, or else either two or three dimensional. In the three dimensional case, the ovoid is directly related to an inversive plane, in much the same way that an ovoid in a finite projective space is.

1. INTRODUCTION

Tallini [3] showed that a finite uniform planar space with an ovoid must be a three-dimensional projective space with an ovoid, using a previous result (Tallini [4]) characterising projective spaces with ovoids. In this paper I have attempted to reach a similar result for linear spaces, with, of course, a different definition of an ovoid - Tallini's can only be applied to a planar space.

Whilst I have not attained an exact duplicate of Tallini's result for linear spaces, I have gone a long way towards it. A projective space with an ovoid gives rise to an inversive plane, and I have managed to show that a finite uniform linear space with an ovoid must be one of

(i) a finite set, where the lines are the pairs of points, and the ovoid is also a pair of points;
(ii) a projective plane with an oval; or
(iii) a three-dimensional linear space with an ovoid, which gives rise to an inversive plane.
In (iii), because of results of Dembowski and Hughes [2] and Thas [5] about inversive planes, if the space is of even order, then it is a projective space.

It should be noted that my result merely parallels Tallini's work; the difference in the definition of an ovoid is such that, if mine were applied to a planar space, the result comparable to Tallini's would be trivial.

2. DEFINITIONS

A linear space $S$ is a non-empty set whose elements are called points, provided with a family of distinguished subsets called lines, such that any two distinct points are in exactly one line. A finite linear space in which each line has the same number of points will be called uniform, although it should be recognised that finite uniform linear spaces are usually called Steiner systems.

A subset $S'$ of a linear space is a subspace if, for any two points of $S'$, all points of the line containing the two are also in $S'$. A hyperplane (or prime) is a proper subspace which meets every line - so that any line is either a subset of a hyperplane, or meets it in exactly one point.

A subset $O$ of a linear space is a cap (or an $h$-cap, where $h = |O|$) if no three of its points are on the same line. A line is a secant, a tangent, or an external line to a cap if it meets it in, respectively, two, one, or no points. For any point $P$ of a cap, the set $Y_P$, the union of all tangents at $P$, is the tangent space at $P$ to the cap. A cap is an ovoid if all its tangent spaces are hyperplanes.

From this point on, $S$ will be considered to be a finite uniform linear space containing an ovoid $O$. In order to remain comparable to Tallini [3], I shall use the following symbols:

- $V$ the number of points in $S$
- $k (= q+1)$ the number of points on a line
- $R$ the number of lines through a point
- $s$ the number of tangents through a point of $O$
- $s_T$ the number of secants through a point $T$ not in $O$
- $t_T$ the number of tangents through $T$