REFORMULATION OF THE BROUWER GEOPOTENTIAL THEORY FOR IMPROVED COMPUTATIONAL EFFICIENCY

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Abstract. The theory, as derived by Brouwer and later modified by Lyddane, of the motion of an artificial Earth satellite perturbed by the first five zonal harmonics is reformulated in terms of an alternate set of variables. This alternate set of variables produces an equivalent solution, has no small eccentricity or small inclination restrictions, and allows calculation of position and velocity with considerably fewer algebraic and trigonometric operations. In addition, the alternate set of variables avoids one solution of Kepler's equation.

1. Introduction

A description of the motion of an artificial Earth satellite perturbed by the zonal harmonics $J_2, J_3, J_4,$ and $J_5$ has been given by Brouwer (1959). He used the von Zeipel method to find a transformation of variables which significantly simplifies the transformed differential equations. Brouwer then found a solution of the transformed equations which, along with the transformation cited above, gives a solution containing all periodic terms through first order and all secular terms through second order, where $J_2$ is assumed to be first order and the remaining zonal harmonics are assumed to be second order. The theory is not applicable for small values of eccentricity or inclination or near the critical inclination.

Lyddane (1963) was able to reformulate the Brouwer transformation* in terms of an alternate set of variables such that the new transformation is also valid for small eccentricities and inclinations. Lyddane gives the transformation

\[
\begin{align*}
  x_1 &= x'_1 + \delta x_1 \\
  x_2 &= x'_2 + \delta x_2 \\
  x_3 &= x'_3 + \delta x_3 \\
  x_4 &= x'_4 + \delta x_4 \\
  x_5 &= x'_5 + \delta x_5 \\
  x_6 &= x'_6 + \delta x_6
\end{align*}
\]  

(1)

where

\[
\begin{align*}
  x'_1 &= a \\
  x'_2 &= e \sin M \\
  x'_3 &= e \cos M \\
  x'_4 &= \sin \frac{I}{2} \sin \Omega \\
  x'_5 &= \sin \frac{I}{2} \cos \Omega \\
  x'_6 &= M + \omega + \Omega
\end{align*}
\]  

(2)

with $a =$ semimajor axis, $e =$ eccentricity, $I =$ inclination, $M =$ mean anomaly, $\omega =$ argument of perigee, $\Omega =$ longitude of ascending node. The $x'_i$ have the same

* The Lyddane modification of the Brouwer theory will be referred to as the Brouwer–Lyddane theory.
definitions as given by Equations (2) except that the quantities on the right-hand side should be replaced by the Brouwer double-primed elements* \( a'', e'', l'', M'', \omega'', \Omega'' \). Lyddane gives the first-order formulas

\[
\begin{align*}
\delta x_1 &= \delta a \\
\delta x_2 &= \delta e \sin M'' + e'' \delta M \cos M'' \\
\delta x_3 &= \delta e \cos M'' - e'' \delta M \sin M'' \\
\delta x_4 &= \frac{1}{2} \delta l \cos l''/2 \sin \Omega'' + \sin l''/2 \delta \Omega \cos \Omega'' \\
\delta x_5 &= \frac{1}{2} \delta l \cos l''/2 \cos \Omega'' - \sin l''/2 \delta \Omega \sin \Omega'' \\
\delta x_6 &= \delta M + \delta \omega + \delta \Omega
\end{align*}
\]

where \( \delta a, \delta e, \delta l, \delta M, \delta \omega, \) and \( \delta \Omega \) are the terms of the Brouwer transformation which, as Lyddane points out, should be treated as the sum of the short- and long-period variations and strictly as functions of double-primed variables rather than the mixture which Brouwer uses.

The Brouwer–Lyddane theory forms the basis for many daily satellite tracking and prediction operations. To predict Cartesian position and velocity with the Brouwer–Lyddane theory, one must:

1. Solve Kepler's equation to find the double-primed true anomaly corresponding to the double-primed mean anomaly;
2. Apply the transformation given in Equations (1);
3. Inversely solve Equations (2) for the osculating orbital elements;
4. Solve Kepler's equation to find the osculating true anomaly corresponding to the osculating mean anomaly; and
5. Compute position and velocity from the osculating orbital elements.

In many cases, operational considerations call for minimizing computer run-time. By defining an alternate set of variables, hereafter called position elements, we have found a convenient reformulation of the Brouwer–Lyddane transformation which avoids the second solution of Kepler's equation and the inverse solution of Equations (2) while still having no small eccentricity or small inclination restrictions. In addition, the position elements transformation requires only two trigonometric operations on the osculating variables in order to compute Cartesian position and velocity. Finally, the number of terms to be computed for the position elements transformation is smaller. In all, this formulation yields considerable computer run-time savings.

* The Brouwer formulas for predicting double-primed elements are given in the Appendix.