NUMERICAL STABILIZATION OF KEPLERIAN MOTION

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Abstract. The time transformation \( \frac{dt}{ds} = r^\alpha \) is studied in detail and numerically stabilized differential equations are obtained for \( \alpha = 1, 2, \) and \( \frac{3}{2} \). The case \( \alpha = 1 \) corresponds to Baumgarte's results.

1. Introduction

The idea of numerical stabilization is to make the mean frequency (mean motion) independent of the initial conditions. In this way, if the problem is orbitally stable according to Poincaré, it becomes stable also in Liaponov sense. From Kepler's equation, it is clear that mean motion depends on the semi-major axis which in turn depends on the energy. The method of fixing the energy during numerical integration to accomplish stability is discussed by Nacozy (1971) and Baumgarte (1972). Nacozy uses a discrete process which brings the numerical solution back to the energy manifold. Baumgarte uses a continuous process by adding control terms to differential equations in such a way that the change in the energy due to numerical errors remains bounded. In Section 2, the time transformation \( \frac{dt}{ds} = r^\alpha \) is introduced and the mean motion for various \( \alpha \) is found. It is shown that the stabilization may be achieved by fixing the energy only when \( \alpha = 0 \) or 1. The stabilizing device is different for other values of \( \alpha \). In Section 3, following Baumgarte (1972) the additional control terms are introduced for \( \alpha = 1, 2, \) and \( \frac{3}{2} \) and their effect is shown. Finally, in the last section, the concept of the time element is discussed. This idea was treated by Stiefel and Scheifele (1971), Baumgarte and Stiefel (1974a, b), Zare and Szebehely (1975), and Nacozy (1981).

2. Mean Motion

From Kepler's equation we have

\[
\frac{dt}{dE} = \frac{a^{3/2}}{\sqrt{\mu}}(1 - e \cos E),
\]

where \( a, e, E \) and \( \mu \) are the semi-major axis, eccentricity, eccentric anomaly and the gravitational constant, respectively.

The independent variable is transformed according to

\[
\frac{dt}{ds} = r^\alpha,
\]

where \( s \) is the new independent variable and \( \alpha \) is a real number.
The period, $P_\alpha$ in the system $s$, is

$$P_\alpha = \frac{2a^{3/2-\alpha}}{\sqrt{\mu}} \int_0^\pi (1 - e \cos E)^{1-\alpha} dE. \quad (3)$$

We define the mean motion by

$$n_\alpha = \frac{2\pi}{P_\alpha}. \quad (4)$$

For the cases of interest: $\alpha = 0; 1; \frac{3}{2};$ and 2 we have

$$n_0 = \frac{\mu}{\sqrt{a^3}}, \quad n_1 = \frac{\mu}{a^3}, \quad n_{3/2} = \frac{\pi \sqrt{\mu(1 + e)}}{2K(k)};$$

$$n_2 = \sqrt{\mu a(1 - e^2)}, \quad (5)$$

where $K(k)$ is the complete elliptic integral of the first kind with $k = \sqrt{2e/(1 + e)}$.

The integrals of the two-body problem are

$$h = -\frac{\mu}{2a} \text{(energy)}, \quad (6)$$

and

$$c^2 = \mu a(1 - e^2) \quad \text{(angular momentum)}. \quad (7)$$

The following remarks might be of interest:

(i) The mean motion depends only on the semi-major axis if $\alpha = 0$ or if $\alpha = 1$; consequently the mean motion depends on the energy, $h$.

(ii) The mean motion depends only on the eccentricity if and only if $\alpha = \frac{3}{2}$; consequently the mean motion depends on the quantity $(c^2h)$.

(iii) The mean motion depends on both the semi-major axis and on the eccentricity if $\alpha = 2$, consequently the mean motion depends on the angular momentum, $c$.

In the Appendix, the case of $\alpha = \frac{3}{2}$ is studied with a different approach.

### 3. Numerical Stabilization

The equation of motion for the two-body problem is

$$\ddot{x} = -\frac{\mu}{r^3} x, \quad (8)$$

where $x$ is the relative position vector, and $r = |x|$.

Following Baumgarte (1972), additional control terms are introduced in the differential equation and their effects are studied in detail. The equation of motion using transformation (2) is

$$x'' = \frac{\alpha}{r^2}(x \cdot x')x' + \mu r^{2\alpha-3} x = 0. \quad (9)$$