A COMPARISON OF THE BOHLIN-VON ZEIPEL AND
BOHLIN-LIE SERIES METHODS IN RESONANT SYSTEMS

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Abstract. Whereas the Bohlin-von Zeipel procedure can be used successfully to construct formal solutions to some resonant dynamical systems, it is shown here that a direct Bohlin–Lie series approach seems not to be feasible. The fact that certain terms lose an order of magnitude on differentiation with respect to the momentum variable leads to a situation which precludes an accurate construction of the first-order term in the generating function. A simple remedy to this impasse is suggested, with particular reference to the Ideal Resonance Problem.

1. Introduction

In his *Méthodes Nouvelles de la Mécanique Céleste*, Poincaré (1893) describes techniques to 'eliminate' successively non-resonant periodic terms from the Hamiltonian of a dynamical system. Each elimination is achieved by means of a canonical transformation of variables, which is constructed using a generating function depending upon the old angle variables and the new momentum variables. After all the periodic terms have been removed in this way, the final Hamiltonian is purely secular. The Poincaré–von Zeipel procedure, usually attributed to von Zeipel (1916), enables the elimination of all the periodic terms to be carried out in one operation. In either case the final Hamiltonian system is very simply integrated and the original variables recovered, through the generating function (s), using a form of Lagrange's implicit function theorem. It is sometimes convenient to separate the given problem into long-period and short-period parts, and possibly parts of intermediate period. Separations of this nature are described in Brouwer (1959) and Garfinkel (1959).

More recently Hori (1966) and Deprit (1969) show that canonical transformations based on Lie series provide a very powerful tool for solving problems of the form outlined in the preceding paragraph. The important properties of the Lie series methods are usually stated as follows: (i) The Poisson brackets, which play a fundamental role in the methods, are canonically invariant, and (ii) the Lie generators depend only upon the new (or the old) variables. The latter property leads to relations between the old and new variables which are in explicit, rather than implicit form. These properties give the Lie series methods a great advantage over the Poincaré–von Zeipel method, for the former are much more readily programmable for a machine.

It is well-known that the Poincaré–von Zeipel and Lie series methods are essentially equivalent, and to the first-order in the small parameter the generators of the two transformations are functionally identical. The relations between the higher-order terms of the generators are given by Shniad (1970).
All the above remarks are applicable to non-resonant dynamical systems. If the system admits a single deep resonance, then one or more of the periodic terms in the original Hamiltonian has a critical argument. To avoid the appearance of a critical divisor in the formal solution the Poincaré–von Zeipel method has to be modified in accordance with the technique attributed to Bohlin (1889). The critical inclination problem in artificial satellite theory is a problem of this type and solutions have been given by, amongst others, Hori (1960) and Garfinkel (1973); the latter being the first global solution. The more general Ideal Resonance Problem, first formulated by Garfinkel (1966) has been investigated in a series of papers, culminating in the global second-order solution (Garfinkel and Williams, 1974). In each of the cited papers the Bohlin–von Zeipel procedure is applied directly to the original Hamiltonian. There are two important features of this procedure: (i) The generating function is no longer purely periodic, and (ii) differentiation with respect to the momentum variable leads to a loss of an order of magnitude in some terms. This latter property, first pointed out by Jupp (1970), and recently remarked upon by Ferraz–Mello (1978), leads to the fact that the generator and new Hamiltonian must be constructed to orders $n + 1$ and $n + 2$ respectively in the small parameter in order to obtain the formal solution to order $n$. To the knowledge of this author no previous attempt, of the kind outlined in Section 3.2, has been made to solve a resonance problem. Here, the Bohlin and Lie Series procedures are applied simultaneously. The outcome, however, is quite unsatisfactory, in that it does not appear to be possible to construct accurately the first order part of the Lie generator. A solution of the Ideal Resonance Problem using Lie Series has been published (Jupp, 1972), but there the Lie Series are used after a preliminary transformation has essentially removed the resonant feature of the original Hamiltonian. This device is outlined in Section 4.

For systems admitting two or more deep resonance the difficulties are very much greater. No method is known to this author which resolves all the difficulties. The Trojan asteroid problem falls into this category and the most complete formal solution is given by Garfinkel (1978). His solution is of a local character, in that it contains a divisor which is critical in certain regions of the solution space.

2. The Non-Resonant Problem

For the purposes of this article we shall consider Hamiltonian systems with a single degree of freedom. The arguments set out in this section can very readily be extended to non-resonant systems with several degrees of freedom.

Consider a system with Hamiltonian
\[ F = F_0(y) + \mu^2 F_2(x, y) + \mu^4 F_4(x, y) + \ldots \] (1)
in which $\mu^2 \ll 1$ and $F_i (i \geq 2)$ is periodic in the coordinate $x$, with period $2\pi$. The equations of motion are
\[ \frac{dx}{dt} = \frac{\partial F}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial F}{\partial x}. \] (2)