ON A PERTURBATION THEORY USING LIE TRANSFORMS

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Abstract. Kamel has recently extended to non-Hamiltonian equations a perturbation theory using Lie transforms. We show here how Kamel's extension can be approached from an intrinsic viewpoint, which reformulation leads to a simpler algorithm. Then we complete Kamel's contribution by establishing the rules for inverting the transformation generated by the perturbation theory, and for composing two such transformations.

1. Introduction

Transformations of coordinates expanded in power series of a small parameter play an important role in the theory of perturbation. Recently a new algorithm has been proposed by Deprit (1969) to build recursively canonical transformations depending on a small parameter. This algorithm is based on the consideration of Lie series and represent a decisive improvement over previous methods. Indeed the transformation is built recursively in explicit form and the Commutation Theorem of Lie series (Gröbner, 1960) implies that the transformation of a function into the new coordinates is a straightforward application of the same algorithm used in defining the transformation. More recently, Kamel (1969b) has proposed a generalization of Deprit's technique to be applied to non-Hamiltonian systems of differential equations.

In the present communication we offer a new presentation of Kamel's algorithm and derive some new formulas related to Lie transforms.

Let us consider the transformation \( x \to y \) from the real \( n \)-vector \( x \) to the real \( n \)-vector \( y \)

\[
x = X(y, \varepsilon),
\]

defined as being the solution at time \( \varepsilon \) of the \( n \)-dimensional system of differential equations

\[
\frac{dx}{d\varepsilon} = W(x, \varepsilon),
\]

for the initial condition \( x(0) = y \). Unless otherwise stated latin letters stand for \( n \)-vectors throughout these notes. The solution (1) can be expanded as a power series of \( \varepsilon \) by the rule

\[
x = \sum_{i \geq 0} \frac{\varepsilon^i}{i!} [D^i x]_{x = y; \varepsilon = 0},
\]

where the differential operator \( D \) is defined by

\[
Df(x, \varepsilon) = \frac{\partial f}{\partial \varepsilon} + \left( \frac{\partial f}{\partial x} \right) \cdot W,
\]

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and can be applied on a scalar as well as on a vector. When \( f \) is a vector, the symbol \( \frac{\partial f}{\partial x} \) stands for the Jacobian matrix and \( \frac{\partial f}{\partial x} \cdot W \) is the image of the vector \( W \) by the matrix \( \frac{\partial f}{\partial x} \). When \( f \) is a scalar, the symbol \( \frac{\partial f}{\partial x} \) stands for the gradient of \( f \) and \( \frac{\partial f}{\partial x} \cdot W \) is the dot product of two vectors.

The **commutation theorem** of Lie series (Gröbner, 1960) which states that

\[
\mathcal{F}(X(y, \varepsilon), \varepsilon) = \sum_{i \geq 0} \frac{\varepsilon^i}{i!} [D^i \mathcal{F}]_{x=y; \varepsilon=0},
\]

shows that the transformation of any function of the vector \( x \) into a function of the vector \( y \) results from the application of the same algorithm used in expanding the transformation (1) itself.

If we wish to generate a canonical transformation it is sufficient to define the system of differential equations (2) as a Hamiltonian system and thus the vector \( W \) as the product of the sympletic matrix \( \mathcal{F} \) by the gradient of a Hamiltonian function \( \mathcal{H}(x, \varepsilon) \)

\[
W(x, \varepsilon) = \mathcal{F} \mathcal{H}(x, \varepsilon).
\]

In that case the differential operator \( D \) applied to a scalar function \( \phi(x, \varepsilon) \) can be rewritten as

\[
D\phi(x, \varepsilon) = \frac{\partial \phi}{\partial \varepsilon} + (\phi; \mathcal{H}),
\]

where the second term of the right hand member is the Poisson bracket of \( \phi \) and \( \mathcal{H} \) in the phase space \( x \).

As a matter of fact Hori (1966, 1967) has also proposed a similar technique to generate canonical transformations. But his definition involves only conservative Hamiltonian systems. Accordingly his system (2) of differential equations reads

\[
dx/d\varepsilon = W(x, \mu),
\]

and the general solution (1) becomes

\[
x = X'(y, \mu, \varepsilon).
\]

Eventually a canonical transformation depending on one small parameter is achieved by setting \( \mu=\varepsilon \). Because the system \( (2') \) has been made artificially conservative by introducing a second parameter \( \mu \), Hori's method is not equivalent to Deprit's technique; more specifically, with the same generating function \( \mathcal{H} \), the two techniques generate different canonical transformations

\[
X(y, \varepsilon) \neq X'(y, \varepsilon, \varepsilon).
\]

This may explain the discrepancies between perturbation equations developed by Hori and Deprit's recursive formulae.

However, it should be noted that Deprit's technique has a marked advantage over Hori's method in that it produces a **recursive algorithm** generating directly the