AN ANALYTIC METHOD TO ACCOUNT FOR DRAG 
IN THE VINTI SATELLITE THEORY

J. S. WATSON, G. D. MISTRETTA, and N. L. BONAVITO
Goddard Space Flight Center, Greenbelt, Md., U.S.A.

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Abstract. In order to retain separability in the Vinti theory of Earth satellite motion when a non-
conservative force such as air drag is considered, a set of variational equations for the orbital elements
are introduced, and expressed as functions of the transverse, radial, and normal components of the non-
conservative forces acting on the system. In this approach, the Hamiltonian is preserved in form, and
remains the total energy, but the initial or boundary conditions and hence the Jacobi constants of the
motion advance with time through the variational equations. In particular, the atmospheric density
profile is written as a 'fitted' exponential function of the eccentric anomaly, which adheres to tabular
data at all altitudes and simultaneously reduces the variational equations to definite integrals with
closed form evaluations, whose limits are in terms of the eccentric anomaly. The values of the limits
for any arbitrary time interval are obtained from the Vinti program.

Results of this technique for the case of the intense air drag satellites San Marco-2 and Air Force
Cannonball are given. These results indicate that the satellite ephemerides produced by this theory in
conjunction with the Vinti program are of very high accuracy. In addition, since the program is
terribly analytic, several months of ephemerides can be obtained within a few seconds of computer
time.

1. Introduction

Vinti (1973) has shown that if a satellite orbit is described by means of osculating
Jacobi $\alpha$'s and $\beta$'s of a separable problem, then a perturbing force $\vec{F}$ makes them vary
according to

$$\dot{\alpha}_K = \vec{F} \cdot \partial \vec{r} / \partial \beta_K, \quad \dot{\beta}_K = -\vec{F} \cdot \partial \vec{r} / \partial \alpha_K, \quad (K = 1, 2, 3).$$

Here $\vec{r}$ is the position vector of the satellite and $\vec{F}$ is any perturbing force, conservative
or non-conservative. If $\vec{F}$ is the force of air drag, the interaction of drag with oblateness
makes it desirable to obtain variations to the order drag $\times J_2$, where $J_2$ is the coefficient
of the second zonal harmonic of the Earth's gravitational potential. The physical
reason for carrying these derivatives through order $J_2$ is the strong variation of drag
with perigee height. In the present paper, we have been able to account for this effect
without introducing the $J_2$ terms into $\partial \vec{r} / \partial \beta_K$ and $\partial \vec{r} / \partial \alpha_K$. The logic behind our
approach requires a rather careful exposition which we shall go into in detail in Section
2. The essence of the method is that for a given time interval, one always does both a
drag free calculation, and an oblateness free calculation with drag, and that these
calculations are done in a self-consistent iterative manner such that the mean
orbital elements never go far astray. The appropriate criterion, to make sure that the
drag-oblateness interaction is being properly accounted for, is that the perigee height
corresponding to initial and final orbital elements of a given interval shall not change
by more than some predetermined amount.
The complexity of those papers which attempt to handle the oblateness-drag interaction in a straightforward manner (Brouwer and Hori, 1961; Sherrill, 1966) illustrates the desirability of finding a new approach. That is the purpose of this paper.

2. Statement of the Problem

In this paper we consider the motion of an artificial Earth satellite in the presence of air drag and the Earth’s gravitational potential. In contrast to the classical methods of numerical integration, our approach will be to present a quadrature algorithm employing analytical expressions for the variation of orbital elements produced by air drag. These expressions are well-defined over expanded subintervals of the solution, and produce accurate agreement with profiles of tabular density. This procedure then allows a flexibility in the selection of end points of the subintervals, which in turn insures a minimum error bound on the required analytical function. In this method the effect of oblateness is accounted for by the Vinti Spheroidal Theory (Vinti, 1969). The changes due to atmospheric resistance for a nonrotating sphere are accounted for by the solutions of the variational equations without oblateness (Sterne, 1960).

Normally, one would wish to represent the variation of atmospheric density by an exponential whose power is a function of the difference between the satellite height and the altitude at a predetermined density (King-Hele, 1964). Such a representation is usually valid only in a neighborhood of this boundary value. The neighborhood or region over which this density representation is in agreement with tabular data such as provided by the *U.S. Standard Atmosphere Supplements*, (1966), is one in which the density scale height is observed to vary in an approximate linear fashion. Throughout our calculations a set of such regions is chosen to meet this requirement. In addition, the initial or boundary value of the atmospheric density for each of these regions is also supplied by the Supplements. Such a representation for atmospheric density would, except under special conditions, exclude closed form integration of the variational equations. To avoid this difficulty, we approximate the atmospheric density variation by an expression which is made to adhere closely to the numerical values of the aforementioned model. By adjusting or advancing boundary conditions over several selected arcs or layers of atmospheric density, we produce a profile that closely agrees with the tabulated data for all heights. These density variation profiles then are held fixed until such time as the perigee height changes by some predetermined amount. At this point the boundary conditions are redetermined over the first half revolution away from perigee, corresponding to a chosen epoch. This again produces a total density profile that is in close agreement with the values from the Supplement tables. Experience indicates that given a criterion of one kilometer change in perigee, this redetermination is not necessary for nearly two months in the cases of the San Marco-2 and Cannonball satellites, but becomes more frequent near the end of the lifetime of each spacecraft.