Abstract. The concept of a Lie series is enlarged to encompass the cases where the generating function itself depends explicitly on the small parameter. Lie transforms define naturally a class of canonical mappings in the form of power series in the small parameter. The formalism generates nonconservative as well as conservative transformations. Perturbation theories based on it offer three substantial advantages: they yield the transformation of state variables in an explicit form; in a function of the original variables, substitution of the new variables consists simply of an iterative procedure involving only explicit chains of Poisson brackets; the inverse transformation can be built the same way.

1. Introduction

Canonical transformations developed in formal series

\[ x = y + \sum_{n \geq 1} \frac{1}{n!} \varepsilon^n y^{(n)}(y, Y; t), \]

\[ X = Y + \sum_{n \geq 1} \frac{1}{n!} \varepsilon^n Y^{(n)}(y, Y; t) \]

in the powers of a small parameter \( \varepsilon \) play an important role in the quantitative as well as qualitative study of dynamical systems described by Hamiltonians of the type

\[ \mathcal{H} \equiv \mathcal{H}(x, X; t; \varepsilon) = \sum_{n \geq 0} \frac{1}{n!} \varepsilon^n \mathcal{H}_n(x, X; t). \]

When the transformations are conservative, the so-called von Zeipel's method proposes to build them from generating functions in mixed variables, like

\[ W \equiv W(x, Y; \varepsilon) = (x | Y) + \sum_{n \geq 1} \frac{1}{n!} \varepsilon^n W_n(x, Y), \]

by the implicit equations

\[ X \equiv X(x, Y; \varepsilon) = W_x, \]

\[ y \equiv y(x, Y; \varepsilon) = W_Y. \]

(N.B. \((x | Y)\) denotes the dot product of the configuration vector \(x\) and the moment vector \(Y\); \(W_x\) indicates the gradient of \(W\) in the configuration space \(x\), a vector made of the partial derivatives of \(W\) with respect to the components of \(x\); \(W_Y\) has a similar interpretation.)

Von Zeipel's method presents serious inconveniences:
(a) An explicit formulation of the transformation and of its inverse requires inverting the Equations (4) to obtain the configuration vector $x$ as a function of the state variables $(y, Y)$, whereafter the result of the inversion is substituted into the Equations (3) to express explicitly the moment vector $X$ in terms of the state variables $(y, Y)$.

From a practical viewpoint, inversions and substitutions are cumbersome operations. Oftentimes the authors of a perturbation theory are satisfied with suggesting a Lagrange series extended to several variables in order to invert Equation (4), but they do not enter into the details of the operations. As a result, their theory is left unfinished, precisely on a point to which practitioners ought to pay special care. For instance, ephemerides of an artificial satellite based on a theory elaborated by means of a von Zeipel's technique may suffer in accuracy from the fact that Delaunay's elements averaged over an anomalistic period are not explicitly developed in terms of the initial conditions (see Lyddane and Cohen, 1962).

(b) The implicit relations (3) and (4) do not lead to a general algorithm to convert into the state variables $(y, Y)$ a function

$$f(x, X; \varepsilon) = \sum_{n \geq 0} \frac{1}{n!} \varepsilon^n f_n(x, X)$$

of the original state variables $(x, X)$. But these conversions have a considerable importance in practice. In the sequence of computations needed to establish and differentially correct the orbit of an artificial satellite, substantial savings would be made if the perturbation theories would express explicitly state functions like longitude, latitude, range and range rate in terms of the averaged elements.

(c) In numerous problems, a transformation to be appropriate must satisfy, besides the criterion of canonicity, other essential conditions, like being analytical in the moments $Y$ or defining the state variables $(x, X)$ as averages of their transformed ones $(y, Y)$. Some of these properties easily transfer from the mapping to the generating function. But this is not the general case. How, for instance, do we select those functions $W$ such that, after inversion and substitution, the coefficients $y^{(n)}$ and $Y^{(n)}$ be analytical in the moment vector $Y$? The problem is a real one. According to Barton (1967), it is a major obstacle to implementing by computer a Delaunay operation.

We felt the shortcomings of von Zeipel's method when we attempted to apply it to normalize a Hamiltonian system in the neighborhood of an equilibrium (Deprit, 1966). As a result of this failure, we found a way of constructing the transformation (1) in a straightforward recursion based on an extension to quasi-periodic solutions of Poincaré's continuation of periodic orbits (Deprit and Henrard, 1967). However, we still failed to give a good answer to the problem of converting in the normalizing variables a function of the primitive state variables $(x, X)$.

In the present communication, we offer a Theory of Transformations that, in our opinion, accomplishes everything that von Zeipel's method does in a general perturbation theory, and yet remedies its shortcomings and inconveniences.