Abstract. The Peano-Baker method is applied to the integration of the variational equations to produce the partial derivatives used in satellite navigation. In this method the analytic form of the state transition partial derivatives can be factored so that numerical integration is applied only to the departures from a simplified analytical model.

The advantage of using the Peano-Baker approach rather than direct integration of the variational equations is that with the Peano-Baker method numerical integration can be performed adequately with low order formulae and relatively large step sizes. Numerical results are indicated.

1. Introduction

The computation of partial derivatives is one of the most important ancillary tasks associated with trajectory estimation. An analytic formulation of position and velocity initial condition partial derivatives for any type of Keplerian motion exists (Goodyear, 1965); however, the forcing functions usually encountered do not permit ready analytic solutions (Pines, 1968). Direct integration of the variational equations is time consuming and usually requires high order formulae and small integration steps. The technique of recursive power series has also been applied to computing the partial derivatives (Schanzle, 1967).

Another method of computing these partial derivatives can be developed based on the Peano-Baker method (Ince, 1926) of solving a system of linear differential equations. This method was first applied to error analysis in celestial mechanics by Myachin (1959) and more recently to the methods of general and special perturbations by Danby (1962a, 1964).

In Section 2 the essentials of the Peano-Baker method are sketched. In Section 3 this method is extended to the forms of equations encountered in celestial mechanics and is applied to computing the partial derivatives of the position and velocity vectors with respect to any parameter in the model (theory of motion) and the Jacobian matrix of the rectangular elements with respect to initial conditions. In Section 4 some numerical results of utilizing the modified Peano-Baker method for a close earth satellite are presented.

2. Peano-Baker Method

Notation: Capital letters of any alphabet denote matrices, lower case letters of the Roman alphabet denote column vectors, either with a subscript denotes a component or element, lower case Greek letters denote scalars, the superscript $T$ denotes the
matrix or vector transpose, superscribed dots denote the time derivatives. Exceptions: 
t is the scalar time and the subscript 0 denotes the initial value.

Consider the system of differential equations
\[ \dot{y} = Uy \]  
where \( U = U(t) \) and at \( t = t_0 \), \( y = y_0 \). If we can find a matrix
\[ \Omega = \Omega(t, t_0; U) \]  
where \( \Omega(t_0, t_0; U) = I \) and
\[ \frac{d\Omega}{dt} = U\Omega \]  
then the solution to Equation (2.1) is
\[ y = \Omega y_0 \quad \text{for} \quad \frac{d}{dt} \left( \Omega \dot{y}_0 \right) = \frac{d\Omega}{dt} \left( y_0 \right) = U \left( \Omega y_0 \right). \]  

Ince (1926) shows that the solution, \( \Omega \), of Equation (2.3) can be obtained by successive approximations as
\[ \Omega(t, t_0; U) = I + \int_{t_0}^{t} U(\tau) d\tau + \int_{t_0}^{t} U(\sigma) \int_{t_0}^{\tau} U(\tau) d\tau d\sigma + \cdots . \]  
The series on the right of Equation (2.5) is the integral of the Neuman series and is called the matrizant or matricant. Ince (1926) also shows that the elements of \( \Omega \) converge absolutely and uniformly throughout any interval containing \( t_0 \) and wholly within the Mittag-Leffler star (i.e. well away from the singular points of \( t \)).

Ince (1926), Frazer et al. (1963), and Gantmacher (1960) develop the theorems: For
\[ \dot{y} = Uy + z(t) \]  
the solution is
\[ y = \Omega y_0 + \Omega \int_{t_0}^{t} \Omega^{-1} z(\tau) d\tau, \]  
where \( \Omega \) is the matrizant for the homogeneous equation.
If the system of equations can be written in the form
\[ \frac{dy}{dt} = \dot{y} = (U + V)y \]  
for \( V = V(t) \) then setting
\[ y = \Omega(t, t_0; U)x, \]  
where \( x_0 = y_0 \) yields the differential equation
\[ \dot{x} = \Omega^{-1} V \Omega x, \]