THE QUASI INTEGRALS*

C. MARCHAL
ONERA, 92320 Châtillon, France

Abstract. The usual Von Zeipel transformations of the Hamiltonian Mechanics are presented, they lead to state functions with extremely slow variations: the 'quasi-integrals'.

The Von Zeipel transformations are implicit: an explicit and direct construction of the quasi-integrals is also presented and the quasi-invariance property of these functions is demonstrated.

These results are applied to the problem of the motion of artificial satellites perturbed by the zonal harmonics of the Earth potential, the quasi-integral is given, it is then so constant that the problem can be considered as integrable.

1. Introduction

The integrals of motion are an essential tool in the resolution of problems of differential equations and it is also useful to know state functions with extremely slow variations, we will call them quasi-integrals.

Such quasi-integrals are generally 'without secular variations to all orders' (as for instance the semi-major axes of the planetary orbits of the solar system). Unfortunately it doesn't imply a strict invariance but we will see that it leads sometimes to so slow variations that the problem of interest becomes practically integrable.

2. Quasi-integrals Without Secular Variations to All Orders

Let us consider an ordinary Hamiltonian problem.

State parameters: \( p = (p_1, p_2, \ldots, p_n) \) and \( q = (q_1, q_2, \ldots, q_n) \).

Parameter of description: the time \( t \).

Hamiltonian function: \( H = H(p, q) \).

Equations of motion: \( \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \); \( \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \).

Let us assume that the Hamiltonian \( H \) has the following form:

\( H \) is analytic in terms of \( p \) and \( q \).

\( H \) is periodic with a period \( 2\pi \) in terms of \( q_1 \).

\[ H = J(p_1) + \varepsilon M(p, q), \] the partial derivatives of \( M \) being of the order of \( dJ/dp_1 \) and \( \varepsilon \) being a small constant quantity.


Copyright © 1980 by D. Reidel Publishing Co., Dordrecht, Holland, and Boston, U.S.A.
We will study $H$ for various values of the main infinitely small $\varepsilon$ and we will decompose $M(p, q)$ into $M = K + L$, the function $K$ being the mean value of $M$ with respect to $q_1$:

$$K(p, q) = \frac{1}{2\pi} \int_0^{2\pi} M(p, q) \, dq_1; \quad L(p, q) = M(p, q) - K(p, q). \quad (3)$$

Hence

$$H = J + \varepsilon (K + L); \quad \frac{\partial K}{\partial q_1} = 0; \quad \frac{1}{2\pi} \int_0^{2\pi} L(p, q) \, dq_1 = 0. \quad (4)$$

The function $C(p, q)$ will be uniquely defined by

$$\frac{\partial C}{\partial q_1} = L(p, q); \quad \frac{1}{2\pi} \int_0^{2\pi} C(p, q) \, dq_1 = 0. \quad (5)$$

The system (1) can be simplified by a Von Zeipel transformation (Von Zeipel, 1916–1917) that leads from $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ to $p_2 = (p_{1,2}, \ldots, p_{n,2})$ and $q_2 = (q_{1,2}, \ldots, q_{n,2})$. That transformation is implicitly defined by

$$S = \text{generating function} = S(p, q_2)$$

$$p_{1,2} = \frac{\partial S}{\partial q_{1,2}}; \quad q_i = \frac{\partial S}{\partial p_i}, \quad (6)$$

and the Hamiltonian form of the equations (1) is conserved provided that the new Hamiltonian $H_2(p_2, q_2)$, identical to $H(p, q)$ be expressed in terms of $p_2$ and $q_2$.

Let us choose for instance that

$$S(p, q_2) = p \cdot q_2 + \varepsilon C(p, q_2)/(dJ/dp_1). \quad (7)$$

Hence

$$p_{1,2} = \frac{\partial S}{\partial q_{1,2}} = p_1 + \varepsilon L(p_1, q_{1,2})/(dJ/dp_1) \quad (8)$$

and

$$H(p, q) = J(p_1) + \varepsilon (K + L) = H_2(p_2, q_2) =$$

$$= J(p_{1,2}) + \varepsilon K(p_2, q_2) + O(\varepsilon^2), \quad (9)$$

we will put

$$H_2(p_2, q_2) = J(p_{1,2}) + \varepsilon K_2(p_2, q_2) + \varepsilon^2 L_2(p_2, q_2) \quad (10)$$

with

$$\frac{\partial K_2}{\partial q_{1,2}} = 0; \quad K_2(p_2, q_2) = K(p_2, q_2) + O(\varepsilon) = K(p, q) + O(\varepsilon) \quad (11)$$