A PROOF OF PASCH'S AXIOM IN THE ABSOLUTE THEORY OF ORIENTED PARALLELITY

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Trapezium and Parallelogram Configurations in the theory of oriented parallelity are considered. The Pasch theorem is proved.

INTRODUCTION

As was shown by W. Szmielew in [5], the axiom of Pasch is equivalent in the ordered affine geometry to the property of invariance of the betweenness relation under parallel projection. In this paper we start with a single 4-place relation which we call the relation of oriented parallelity and a theory based on a set of axioms relating to this notion. The four axioms are satisfied in both the affine (euclidean) planes and the hyperbolic (ordered) planes. The theory is called by us "the Absolute Theory of Oriented Parallelity", or, briefly, AOP. The relation of betweenness is a derived notion of the theory, as is also the relation of collinearity. The main result of this paper consists in showing that the axiom of Pasch is a theorem of AOP. Two of the axioms, namely, the Parallelogram axiom (Axiom 8) and the Trapezium axiom (Axiom 9) are crucial.
Though perhaps not very surprising, the results of this paper will help put the axiom of Pasch in a general perspective of parallelity in absolute (oriented) geometry and constitute a generalization of the result of Szmielew quoted above. We have not considered the problem of independence of the axioms of AOP.

The relation of oriented parallelity is an equivalence relation, its equivalence classes will be referred to as ends. One can consider ends as points of the configuration related to the Pasch Theorem, then it turns out that the Parallelogram Axiom and the Trapezium Axiom are special forms of the Pasch Theorem. Also some other theorems of this type are investigated in the paper.

ABSOLUTE THEORY OF ORIENTED PARALLELITY AND PASCH CONFIGURATION

By the absolute theory of oriented parallelity (dimension-free) we mean the theory based on the following axioms:

A0 \( \exists a,b \ [axb] \)

A1 \( ab \parallel cc \)

A2 \( ab \parallel ba \iff a = b \)

A3 \( \neg ab \wedge ab \parallel pq \wedge ab \parallel rs \iff pq \parallel rs \) (transitivity axiom)

A4 \( ab \parallel bc \iff cb \parallel ba \)

A5 \( ab \parallel bc \iff ab \parallel ac \)

A6 \( ab \parallel ac \iff ab \parallel bc \vee ac \parallel cb \)

A7 \( \exists a,b,c,d [\neg ab \parallel cd \wedge \neg ab \parallel dc] \) (lower dimension axiom)

A8 \( \exists d [ab \parallel cd \wedge ac \parallel bd \wedge d \neq b] \) (Parallelogram Axiom)

A9 \( b \neq p \wedge bp \parallel pc \iff \exists d [ap \parallel pd \wedge ab \parallel cd] \) (Trapezium Axiom)

This theory will be denoted by AOP.

As an immediate consequence of the axioms of AOP we get the following proposition.

PROPOSITION 1 (i) \( aa \parallel bc \),

(ii) \( ab \parallel ab \),

(iii) \( ab \parallel cd \iff cd \parallel ab \),

(iv) \( ab \parallel pq \wedge pq \parallel cd \wedge p \neq q \iff ab \parallel cd \).