GENERALIZED DOUBLE INVOLUTIVE GEOMETRY OF THE TRIANGLE

Adolfo Quirós and Pablo Rubio

We study a generalized geometry of the triangle, based on the idea of letting two arbitrary points play the role that the centroid and the orthocenter play classically. We thus generalize some of the classical results and constructions and also prove some new results in ordinary metric geometry.

1. INTRODUCTION.

Some of the classical theorems in the elementary geometry of the triangle, such as the properties of the Euler line and the nine-point (or Feuerbach) circle, involve noticing that there is a certain symmetry between the projective properties of the centroid and the orthocenter of the triangle. Our aim in this paper is to show how one can generalize these properties, and hence obtain a generalized double involutive geometry of the triangle, by letting two (almost) arbitrary points $P$ and $P'$ in the plane play the role of the centroid $G$ and the orthocenter $H$.

The key tool we use in this paper is the notion of the duplicate of a geometric construction. The idea is the following. Suppose we start with an ordered pair of points in the plane $(P, P')$ and some other fixed data such as a reference triangle $ABC$, and we carry out some geometric construction from the data to obtain a new geometric object $M$ (it may be a point, a pair, a line, a conic, etc.). If we now perform the same construction but starting not from the pair $(P, P')$ but from the pair $(P', P)$, we obtain a new result $M'$ that, by definition, is the duplicate of the previous one.

Let us give formal definitions. Let $\mathbb{P}^2$ denote the projective plane. A (geometric) construction will be a function $f$ defined on a symmetric subset $\mathcal{D}$ of $\mathbb{P}^2 \times \mathbb{P}^2$ and whose range is a set of geometric objects of some sort (points, pairs of points, lines, conics,...). Given a construction $f$, we define its duplicate $f'$ to be the construction having the same domain and range as $f$ and such that $f'(P', P) = f(P, P')$ for all $(P, P')$ in $\mathcal{D}$. A construction is called self-duplicate if it is its own duplicate. We will usually abuse the language
and we will speak, for example, of the duplicate of a conic meaning the duplicate of the
construction we use to get the conic.

If we fix a system of coordinates \((X, Y, Z)\) for which \(P = (x, y, z)\) and \(P' = (x', y', z')\) and we let \(M = f(P, P')\) be the result of construction \(f\), \(M\) will be determined by coordinates or equations depending upon \(x, y, z, x', y', z'\). If in these coordinates or equations we carry out the substitutions

\[
\begin{align*}
    x &\mapsto x', y \mapsto y', z \mapsto z' \\
x' &\mapsto x, y' \mapsto y, z' \mapsto z
\end{align*}
\]

we will get the coordinates or equations for the duplicate \(M'\) of \(M\). Hence, (1.1) defines a duplication transformation between constructions, which is clearly involutive since applying (1.1) twice we get the identity. Some other obvious properties of duplication are the following. The duplicate of the pair \((P, P')\) (that is, of the construction \(f(P, P') = (P, P')\)) is the pair \((P', P)\). A point (resp. a curve) in the projective plane is self-duplicate if and only if \((x, y, z)\) and \((x', y', z')\) play a symmetric role in its coordinates (resp. equation). The degree and class of any curve and its duplicate are the same, and therefore they have the same topological and projective properties. Finally, the triangle of reference is self-duplicate, since its vertices are independent of \(P\) and \(P'\).

Despite the abuse of language that we have already mentioned, it is very important to insist that duplication should be applied to constructions, and not to the geometric objects that result from them when we consider a specific pair \((P, P')\). In particular (1.1) should not be seen simply, for example, as a mapping from the plane to itself. Let us give a simple example to show what we mean.

Let the following four points be given as data: \(A = (1, i, 0), B = (1, -i, 0), Q_1 = (1, 0, 0), Q_2 = (0, 0, 1)\). Consider now the following constructions: given a pair of points \((P, P')\) (in a sufficiently general position for everything make sense), let \(C\) be the pencil of conics through \(A, B, P\) and \(P'\), let \(C_1\) be the conic in \(C\) passing through \(Q_1\) and let \(C_2\) the conic in \(C\) passing through \(Q_2\). The three constructions \(C\), \(C_1\) and \(C_2\) are clearly self-duplicate. Since all conics in \(C\) are of the form \(\lambda C_1 + \mu C_2\), one may be tempted to conclude that any conic in \(C\) is self-duplicate, which is not true: it depends on how we construct it! If we let, for example, \(C_3\) be the conic in \(C\) that is tangent to the line \(PQ_1\) (of course it must be tangent at \(P\), it turns out that \(C_3\) is not self-duplicate. This is not a contradiction, the whole point being that the values \(\lambda\) and \(\mu\) that we use to write \(C_3 = \lambda C_1 + \mu C_2\) are functions of \((P, P')\) not invariant under (1.1).

To be more concrete, let \(P = (1, 0, 1), P' = (0, 1, 1)\). Then \(C_1(P, P') = C_1(P', P)\) is given by \(Z^2 - XZ - YZ = 0\), and \(C_2(P, P') = C_2(P', P)\) is given by \(X^2 + Y^2 - XZ - YZ = 0\). One then has \(C_3(P, P') = C_1(P, P') + C_2(P, P')\) while \(C_3(P', P) = C_1(P', P) - C_2(P', P)\). Of course the construction \(C_1 + C_2\) is again self-duplicate, but it does not equal the conic \(C_3\) in all cases, only in some of them.

A word about notation. Most of it is standard. Given points \(A_1, A_2, \ldots, A_n\), we denote by \(A_1A_2 \ldots A_n\) the polygon with the corresponding vertices. \(A_1A_2\) will denote the line through \(A_1\) and \(A_2\) except at some points in Section 6 where it will mean the oriented segment from \(A_1\) to \(A_2\). We will sometimes confuse a curve with its equation, and therefore, given two curves \(C_1\) and \(C_2\) with equations \(f_1 = 0, f_2 = 0\), by \(C_1C_2\) and \(C_1 + C_2\) we will