THE POSSIBLE MOTIONS OF A SATELLITE
ABOUT AN OBLATE PLANET

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Abstract. The regions of motions of a satellite for given values of energy and angular momentum about polar axes are shown. Special attention is paid to the circular equatorial orbits which have been shown to be Hill stable. The anomalistic and the nodal period for the motions near to the circular equatorial orbits have been found.

1. Introduction

The regions of motions for the conservative dynamical systems may be established by a theorem given by Zare (1976). The theorem is applied to the problem of the three bodies by Zare (1976, 1977) and to the problem of a rigid body with a fixed point by Zare and Levinson (1977). The aim of this paper is to apply the theorem to the motions of a satellite about an oblate planet. The problem is formulated in Section 2. The theorem is stated in Section 3 and it is applied to the problem at hand. The circular equatorial orbits emerge as the exact solutions with the desired property of Hill stability. These orbits are discussed in Section 4. Using the rather complicated variation of parameters technique, an expression for the anomalistic period of a satellite was derived by Moe and Karp (1961) and an expression for the nodal period of a circular satellite was derived by Sturms (1962). In Section 5 for the motions near to the circular equatorial orbits, we obtain both the anomalistic and the nodal period.

2. Formulation

In a coordinate system with the equatorial plane of the planet taken as the $x - y$ plane, the potential of the planet is given by

$$V = -\frac{\mu}{r} \left[ 1 + \frac{JR^2}{2r^2} \left( 1 - 3\frac{z^2}{r^2} \right) \right]. \quad (1)$$

where $\mu$ is the gravitational constant of the planet, $R$ is its equatorial radius, $J$ is the coefficient of the second harmonic, and $r$ is the distance from the center of the planet.

The Hamiltonian for a satellite in the gravitational field of the planet is given by

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{\mu}{r} \left[ 1 + \frac{JR^2}{2r^2} \left( 1 - 3\frac{z^2}{r^2} \right) \right]. \quad (2)$$
The equations of motion are
\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial p_x}, \\
\dot{y} &= \frac{\partial H}{\partial p_y}, \\
\dot{z} &= \frac{\partial H}{\partial p_z}, \\
\end{align*}
\]
where
\[
\begin{align*}
\dot{p}_x &= -\frac{\partial H}{\partial x}, \\
\dot{p}_y &= -\frac{\partial H}{\partial y}, \\
\dot{p}_z &= -\frac{\partial H}{\partial z}.
\end{align*}
\]

We perform the extended point transformation defined by
\[
\begin{align*}
x &= \frac{\partial W}{\partial p_x}, \\
y &= \frac{\partial W}{\partial p_y}, \\
z &= \frac{\partial W}{\partial p_z}, \\
p_r &= \frac{\partial W}{\partial r}, \\
p_\theta &= \frac{\partial W}{\partial \theta}, \\
p_\phi &= \frac{\partial W}{\partial \phi},
\end{align*}
\]
where the generating function $W$ is given by
\[
W = p_x r \cos \theta \cos \phi + p_y r \cos \theta \sin \phi + p_z r \sin \theta.
\]
Substituting the new variables for the old variables in the Hamiltonian, it may be shown that $\phi$ is an ignorable coordinate. Consequently the corresponding momentum is an integral of the motion
\[
p_\phi = k = \text{const.}
\]
The Hamiltonian reduces to
\[
H = \frac{1}{2} \left( \frac{p_r^2}{r^2} + \frac{p_\theta^2}{r^2 \cos^2 \theta} + \frac{k^2}{r^2} \right) - \frac{\mu}{r} \left[ 1 + \frac{JR^2}{2r^2} (1 - 3 \sin^2 \theta) \right]
\]
The equations of motion are
\[
\begin{align*}
\dot{\theta} &= \frac{\partial H}{\partial p_\theta}, \\
\dot{r} &= \frac{\partial H}{\partial p_r}, \\
\dot{\phi} &= \frac{\partial H}{\partial k}, \\
\end{align*}
\]
After Equations (8) are integrated, the following equation may be used to determine $\phi$,
\[
\dot{\phi} = \frac{\partial H}{\partial k}.
\]

3. Regions of Motions

Since the Hamiltonian does not depend on the time explicitly, it is an integral of the motion, therefore the equation
\[
H = h = \text{const}
\]