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An arithmetic Riemann–Roch theorem

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Introduction

We prove in this paper an arithmetic analog of the Riemann–Roch–Grothendieck theorem for the determinant of the cohomology of an Hermitian vector bundle of arbitrary rank on a family of arithmetic varieties of arbitrary dimension. We also show that high powers of ample line bundles on arithmetic varieties have small sections.

Let X and Y be regular quasi-projective flat schemes over $\mathbb{Z}$. Consider an Hermitian vector bundle $\bar{E} = (E, h)$ on $X : E$ is an algebraic vector bundle on X and h is an Hermitian metric on the associated holomorphic vector bundle on $X(\mathbb{C})$, which is invariant under complex conjugation. In [GS2] we defined arithmetic Chow groups $\widehat{CH}^p(X)$, $p \geqq 0$, and in [GS3] we attached to (E, h) arithmetic characteristic classes such as the Chern character $\widehat{ch}(E, h) \in \widehat{CH}^{\cdot}(X)_{\mathbb{Q}} = \bigoplus_{p \geqq 0} \widehat{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, and the Todd class $\widehat{Td}(E, h)$. Assume now that $f : X \to Y$ is a smooth projective morphism from X to Y. The determinant of cohomology $\lambda(E) = \det Rf_*(E)$ is an algebraic (graded) line bundle on Y. Choose an Hermitian metric h_f, invariant by conjugation, on the relative tangent space Tf, whose restriction to each fiber of f over $Y(\mathbb{C})$ is Kähler. The line bundle $\lambda(E)$ can then be equipped with the Quillen metric h_Q ([Q2], [BGS1] or 4.1.1 below).

Our main result (Theorem 7) computes the first arithmetic Chern class of $(\lambda(E), h_Q)$ in the $\mathbb{Q}$-vector space $\widehat{CH}^1(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$. It reads

$$(1) \quad \hat{c}_1(\lambda(E), h_Q) = f_*(\widehat{ch}(E, h)\widehat{Td}(Tf, h_f) - a(ch(E_{\mathbb{C}})Td(Tf_{\mathbb{C}})R(Tf_{\mathbb{C}})))^{(1)} .$$

Here $\alpha^{(1)}$ is the component of degree one of $\alpha \in \widehat{CH}^{\cdot}(Y)_{\mathbb{Q}}$, a is the map from the real cohomology of $Y(\mathbb{C})$ to $\widehat{CH}^{\cdot}(Y)_{\mathbb{Q}}$ defined in [GS2, 3.3.4] and in 2.2.1 below, and R is the additive characteristic class (in real cohomology) attached to the power series

$$R(x) = \sum_{\substack{m \text{ odd} \\ m \geqq 1}} \left(2\zeta'(-m) + \left(1 + \frac{1}{2} + \ldots + \frac{1}{m}\right)\zeta(-m) \right) \frac{x^m}{m!}$$

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which we introduced in [GS4] ($\zeta(s)$ is the Riemann zeta function, and $\zeta'(s)$ its derivative).

Formula (1) was conjectured in [GS4, Conjecture 1]. The main step in the proof of this formula consists in factoring the map f as the composition $f = g \circ i$, where $i: X \to P$ is a closed (regular) immersion and $g: P = \mathbb{P}_Y^N \to Y$ is the N-dimensional projective space over Y. Choose a resolution

$$0 \to E_m \to E_{m-1} \to \dots \to E_0 \to i_* E \to 0$$

of the coherent sheaf $i_* E$ on P, and (arbitrary) Hermitian metrics on E_j, $j \geqq 0$, as well as a Kähler metric on Tg. We show that (1) for f and $\bar{E}$ follows from the same identity for g and $\bar{E}_j$, $j \geqq 0$. Indeed, the difference

$$\hat{c}_1(\lambda(E), h_Q) - \sum_{j \geqq 0} (-1)^j \hat{c}_1(\lambda(E_j), h_Q)$$

was computed by Bismut and Lebeau [BL], while the corresponding alternating sum of the right-hand sides in (1) was computed in [BGS3, Theorem 4.13].

We are thus reduced to the case of the projection $g: \mathbb{P}_Y^N \to Y$. When E is the trivial bundle and $Y = \operatorname{Spec}(\mathbb{Z})$, formula (1) was shown in [GS4, Theorem 1]. The general case follows by simple reductions, using the closed immersions $\mathbb{P}^N \to \mathbb{P}^{N+1}$ and the main step above.

This proof of (1) was described in [GS7]. The details are given in paragraphs 4.2.3 and 4.2.4 which, when f is smooth, can be read independently from the rest of the paper.

We also generalize (1) in several ways, in order to allow singularities on the special fibers of X or Y over $\mathbb{Z}$ (this might be of some use, since resolution of singularities is not currently available for schemes of finite type over $\mathbb{Z}$). More specifically, we consider two cases. Case(i): Y is regular, the generic fiber $X_{\mathbb{Q}}$ is smooth, f is projective, and smooth over $X_{\mathbb{Q}}$, and $\mathscr{F}$ is a coherent sheaf on X, which is locally free on $X_{\mathbb{Q}}$ and equipped with an Hermitian metric on $X(\mathbb{C})$. Case(ii): $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ are smooth, f is l.c.i., and $\bar{E}$ is an Hermitian vector bundle on X.

To make sense of a Riemann–Roch–Grothendieck theorem for $\lambda(\mathscr{F})$ in case (i), or $\lambda(E)$ in case (ii), we need to extend our previous constructions in [GS2] and [GS3] to the singular case. So we introduce "homological Chow groups" $\widehat{CH}_{\cdot}(X)$, cap-products between $\widehat{CH}^{\cdot}$ and $\widehat{CH}_{\cdot}$, and more generally some kind of "operational formalism" in the sense of Fulton [Fu2]. In case (ii), the statement (1) becomes an identity in $\widehat{CH}_{\cdot}(Y)_{\mathbb{Q}}$, and $\widehat{Td}(Tf)$ has to be replaced by the arithmetic Todd class of the relative tangent complex to f (see 2.6.2). In Case (i), we define a notion of Chern character with supports, and then a characteristic class $\tau(\bar{\mathscr{F}}) \in \widehat{CH}_{\cdot}(X)_{\mathbb{Q}}$ which takes the place of $\widehat{ch}(E, h)\widehat{Td}(Tf, h_f)$ in formula (1). Our theorem (Theorem 7) is then in the style of the singular Riemann Roch theorem of [BFM]. This requires us to combine the Grassmannian graph construction of [BFM] with the study of complex immersions in [BGS2] and [BGS3].

The plan of this paper is as follows. In Sect. 1 we study the Grassmannian graph construction from the algebraic geometric point of view. In particular we show an interesting rigidity property of this construction (Theorem 2), and deduce from it a technical lemma, to be used in Sect. 3. In Sect. 2 we introduce $\widehat{CH}_{\cdot}$, show some functorial properties of these groups, and define cap products and characteristic