A RUNGE-KUTTA NYSTRÖM ALGORITHM*

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Abstract. A Runge-Kutta algorithm of order five is presented for the solution of the initial value problem where the system of ordinary differential equations is of second order and does not contain the first derivative. The algorithm includes the Fehlberg step control procedure.

1. Introduction

A Runge-Kutta algorithm is presented for the initial value problem described by a system of second order differential equations of the form

\[ x = f(t, x), \quad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0. \]

The terms \( x, \dot{x}, x, f \) are vectors. The first derivative, \( \dot{x} \), does not appear explicitly in the functions \( f \).

The algorithm yields the solution, \( x \) and \( \dot{x} \) of the initial value problem where the independent variable \( t_0 \) has been incremented by the step, \( h \). Runge-Kutta methods which deal directly with the second order differential equations, and thereby increase the computational efficiency by avoiding the customary technique of reducing the equations to a system of first order, were first introduced by Nyström (1925). These methods that treat the second order differential equations directly will be referred to as Runge-Kutta-Nyström (RKN) algorithms. RKN algorithms have been developed by Fehlberg (1964, 1972) which predict the size of \( h \) for the subsequent step by estimating the local truncation error of the solution of \( x \) by evaluating the difference between two solutions of \( x \) that are of different orders of accuracy.

This RKN algorithm utilizes an estimate of the local truncation error of both \( x \) and \( \dot{x} \) for calculating the step size by computing the difference between solutions that are of accuracy of \( O(h^4) \) and \( O(h^5) \). For \( t = t_0 + h \), let the two solutions of accuracy \( O(h^4) \) and \( O(h^5) \) be \( X, \dot{X} \) and \( \hat{X}, \hat{\dot{X}} \), respectively,

\[ X = \hat{X}_0 + X_0 h + h^2 \sum_{\kappa=0}^{3} \hat{C}_\kappa f_\kappa + O(h^5), \]

\[ \dot{X} = \hat{X}_0 + h \sum_{\kappa=0}^{4} \hat{C}_\kappa f_\kappa + O(h^5), \]

and,

\[ \hat{X} = \hat{X}_0 + \hat{X}_0 h + h^2 \sum_{\kappa=0}^{4} \hat{C}_\kappa f_\kappa + O(h^6), \]

\[ \hat{\dot{X}} = \hat{X}_0 + h \sum_{\kappa=0}^{5} \hat{C}_\kappa f_\kappa + O(h^5), \]

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where
\[ f_0 = f(t_0, X_0) \]
\[ f_\kappa = f \left( t_0 + \alpha_\kappa h, X_0 + \hat{X}_0 \alpha_\kappa h + h^{\kappa-1} \sum_{\lambda=0}^{\kappa-1} \gamma_{\kappa \lambda} f_\lambda \right), \]
\[ \kappa = 1, \ldots, 5. \]

The difference between the two solutions gives the estimates of the leading terms of the local truncation errors, \( TEx \) and \( TEx' \) which are of \( O(h^5) \),

\[ TEx = X - \hat{X} \]
\[ TEx' = \hat{X} - \hat{\hat{X}}. \]

The derivation of the equations of condition and their solution for the coefficients \( \alpha, \gamma, C, \hat{C}, \hat{\hat{C}} \) are similar to standard Runge-Kutta developments (Fehlberg, 1972). The coefficients, given in Table I, were a compromise choice between minimum error terms for the \( \hat{X} \) and \( \hat{\hat{X}} \) solution and a reliable estimation technique for the step size. The choice of coefficients have resulted in an RKN algorithm that has proven efficient for a large selection of test problems that are representative of problems encountered in celestial mechanics.

### Table I

<table>
<thead>
<tr>
<th>( \alpha_\kappa )</th>
<th>( \gamma_{\kappa \lambda} )</th>
<th>( C_\kappa )</th>
<th>( \hat{C}_\kappa )</th>
<th>( \hat{\hat{C}}_\kappa )</th>
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<td>( 1 )</td>
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<td>( 0 )</td>
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</tr>
</tbody>
</table>

2. Implementation of the RKN Algorithm

Two choices for the solution as the computation proceeds from step to step are either \( X, \hat{X} \), or \( \hat{X}, \hat{\hat{X}} \). The coefficients have been chosen such that the \( \hat{X}, \hat{\hat{X}} \) solution should be used. It is not necessary to compute \( X, \hat{X} \). Only \( \hat{X}, \hat{\hat{X}} \), and the truncation error terms need to be computed,

\[ TEx = \frac{4}{15} \left( \frac{1}{3} f_0 - f_2 + f_3 - \frac{1}{3} f_4 \right) \]
\[ TEx' = \frac{7}{15} \left( -\frac{1}{6} f_0 + \frac{3}{2} f_2 + f_3 + \frac{3}{2} f_4 - \frac{1}{6} f_5 \right). \]

Since \( \gamma_{5 \lambda} = \hat{C}_\lambda, \lambda = 0, \ldots, 4 \), and since \( \alpha_5 = 1 \), the last function evaluation, \( f_5 \), during a step is identical to the first function evaluation, \( f_0 \), for the next step. After the initial step in the algorithm, the last function evaluation of a step can be used for the first