THE TRANSITION FROM ELLIPTIC TO HYPERBOLIC ORBITS IN THE TWO-BODY PROBLEM BY SLOW LOSS OF MASS

L. VAN DER LAAN and F. VERHULST
Mathematical Institute, University of Utrecht, Utrecht, The Netherlands

(Received 17 March, 1972)

Abstract. Transition from elliptic to hyperbolic orbits in the two-body problem with slowly decreasing mass is investigated by means of asymptotic approximations. Analytical results by Verhulst and Eckhaus are extended to construct approximate solutions for the true anomaly and the eccentricity of the osculating orbit if the initial conditions are nearly-parabolic. It becomes clear that the eccentricity will monotonously increase with time for all mass functions satisfying a Jeans-Eddington relation and even for a larger set of functions. To illustrate these results quantitatively we calculate the eccentricity as a function of time for Jeans-Eddington functions $n = 0(1) 5$ and $18$ nearly-parabolic initial conditions to find that $93$ out of $108$ elliptic orbits become hyperbolic.

1. Introduction

In a recent paper by Verhulst (1972) a number of asymptotic solutions have been given for the two-body problem with slowly decreasing mass according to Jeans’ mode. Moreover a stability analysis was presented for a stationary parabolic solution.

The purpose of this note is to study the transition of elliptic orbits to hyperbolic orbits by the process of slow loss of mass. The notion of slow loss of mass was introduced by Verhulst (1969) by the following requirement for the expansion parameter $\alpha/\beta$:

$$\frac{\alpha}{\beta} \leq 3 \sqrt[3]{3/16},$$

where

$$\beta = \frac{G^2}{c^3}.$$

Here $G$ is the gravitational constant, $c$ the constant of areas. The slow decrease of the total mass $m$ of the two-body system is introduced by putting $m = m(\alpha t)$ where $\alpha$ is a small, positive constant. In a large number of cases the function $m$ is chosen from the set of functions defined by a so-called Jeans-Eddington relation

$$\dot{m} = - \alpha m^n, \quad m(0) = m_0,$$

where $n$ is a real constant, $m_0$ the initial mass of the system. For each $n$ we can obtain a solution of Equation (3), which we call a Jeans-Eddington function $m_n(t)$.

To apply inequality (1) we note that the dimension of $\alpha$ is determined by the choice of the mass function $m(\alpha t)$; e.g. in the case of Jeans-Eddington functions the dimension of $\alpha$ is determined by $\dot{m}/m^n$.

It is clear from the paper by Verhulst (1972) that transition from elliptic to hyperbolic orbits may occur if the initial orbits are nearly-parabolic, so we pay attention...
nearly exclusively to this situation. In selecting our mass functions \( m(\alpha t) \) we shall make use of conclusion (1) of Verhulst (1972), while at the same time we are able to test conclusion (2) of that paper which asserts that orbits cannot change from elliptic to hyperbolic type in the neighbourhood of \( f = \pi \), if the selected function \( m(\alpha t) \) satisfies a certain instability criterion.

It has been concluded by Verhulst (1969, Section 4.3) that no hyperbolic orbits will arise in a finite time if we start with an elliptic orbit for Jeans-Eddington function \( n = 3 \). This conclusion is not correct and we will show in Section 3 that this result is connected with the use of the eccentric anomaly \( E \) for eccentricity \( e < 1 \) instead of \( e < 1 \). It is indeed possible to use \( E \) for \( e \leq 1 \) but the results need careful interpretation. In Sections 4 and 5 the eccentricity and true anomaly \( f \) of the osculating orbit are approximated as functions of time in order to study the possibility of transition from elliptic to hyperbolic orbits in Section 6.

### 2. The Initial Value Problem

The variation with time of the eccentricity and the true anomaly is governed by the equations (see Verhulst, 1972)

\[
\frac{de}{dt} = -(e + \cos f) \frac{\dot{m}}{m} \\
\frac{df}{dt} = \beta (1 + e \cos f)^2 m^2 + \frac{\sin f \dot{m}}{e m}.
\]

The equations can be used for both elliptic and hyperbolic orbits but the initial values \( e_0, f_0 \) are restricted to the nearly-parabolic domain \( D_3 \) given by

\[
e_0 \approx 1 - \frac{1}{2} (\alpha/\beta)^{2/3}.
\]

It is useful to introduce in \( D_3 \) a local variable \( \tilde{e} \) by

\[
1 - e^2 = \tilde{e}^2 (\alpha/\beta)^{2/3}.
\]

If we restrict ourselves to the interval of time

\[
t \ll k/\alpha,
\]

in which the constant \( k \) is determined by the choice of \( m \), we obtain from Verhulst and Eckhaus (1970) the following approximate integral of Equations (4)–(5) in \( D_3 \):

\[
F(\alpha t) \frac{\sin E}{\tilde{e}} = \frac{1}{2} \left( e_0^2 - \tilde{e}^2 \right) + F(0) \frac{\sin E_0}{\tilde{e}_0} + O \left( \left( \frac{\alpha}{\beta} \right)^{2/3} \right).
\]

Here \( O \) is Landau's order symbol; \( F(\alpha t) \) is defined by

\[
F(\alpha t) = - \frac{\dot{m}}{\alpha m^3}.
\]