Abstract. Sufficient conditions are given, which, if fulfilled, enable one to determine, rigorously, that an ejected particle will not escape from a three-body system.

1. Introduction

This paper deals with the gravitational problem of three bodies. Occasionally it is useful to be able to determine, without integrating numerically for an extended period of time, whether an ejected particle will escape or not. Sufficient conditions for escape have been given previously by the author (Standish, 1971, hereafter referred to as Paper 1). Here, sufficient conditions for return are given. The derivation and the notation used here are similar to those in Paper 1.

Let \( r \) be the distance of mass \( m_b \) from mass \( m_a \), and let \( \rho \) be the distance of mass \( m_c \) from the center of mass of \( m_a \) and \( m_b \) (Figure 1). The subscripts are chosen such that

\[
\text{the conditions, } r \leq \rho_{ac} \text{ and } r \leq \rho_{bc}, \text{ are fulfilled. Further, it may be shown that when the total energy, } E, \text{ is negative, the minimum interparticle distance is bounded: } r \leq r_* = \left| \frac{G(m_a m_b + m_b m_c + m_a m_c)}{|E|} \right|.
\]

**Theorem.** If at some time, \( t_0 \),

(i) \( x_0 \equiv r_*/\rho_0 < 1 \),

(ii) \( \dot{\rho}_0 > 0 \), and

(iii) \( \dot{\rho}_0^2/2 < \frac{GM}{\rho_0} \left[ 1 - M_a M_b \frac{x_0^2}{(1 - x_0)} \right] - \frac{Q^2}{\rho_0^2} \),

\( \rho_0 \) is the initial distance of mass \( m_c \) from the center of mass of \( m_a \) and \( m_b \).
then it will be the case that \( \dot{\theta}(t) = 0 \) for some time, \( t \), in the interval, \( t_0 < t < \infty \), where \( M = m_a + m_b + m_c \), \( M_a = m_a / (m_a + m_b) \), \( M_b = m_b / (m_a + m_b) \), and

\[
Q = \frac{M |L|}{m_c (m_a + m_b)} + \left\{ \frac{2GM^2 M_a M_b}{m_c r_*} \times \left[ \frac{M_a M_b}{m_c} (m_a + m_b) + x_0 + M_a M_b \frac{x_0^3}{(1 - x_0)} \right] \right\}^{1/2}.
\]

The total angular momentum, \( L \), and the total energy, \( E \), may be written as follows:

\[
L = \frac{1}{g_1} (\mathbf{r} \times \dot{\mathbf{r}}) + \frac{1}{g_2} (\mathbf{q} \times \dot{\mathbf{q}}), \quad \text{and}
\]

\[
E = \frac{1}{2g_1} \dot{r}^2 + \frac{1}{2g_2} \dot{q}^2 - F,
\]

where \( g_1 = (m_a + m_b)/m_a m_b, g_2 = M/m_c (m_a + m_b) \), and

\[
F = G \left( \frac{m_a m_b}{r} + \frac{m_a m_c}{q_{ac}} + \frac{m_b m_c}{q_{bc}} \right).
\]

**Proof.** The differential equation for \( \theta \) is

\[
\ddot{\theta} = g_2 \left[ \frac{p_{\theta}^2}{\cos^2 \phi \dot{q}^3} + \frac{p_{\dot{\theta}}^2}{\dot{q}^3} \right] + g_2 \frac{\partial F}{\partial q},
\]

\[
= \frac{1}{q^3} |\mathbf{q} \times \dot{\mathbf{q}}|^2 + g_2 \frac{\partial F}{\partial q},
\]

where the spherical coordinates are defined by

\[
\mathbf{q} = q (\cos \theta \cos \phi, \sin \theta \cos \phi, \sin \phi)^T.
\]

It may be shown that

\[
\ddot{\theta} \leq \frac{Q^2}{q^3} + GM \left[ -\frac{1}{r^2} + M_a M_b \sum_{i=2}^{\infty} (i + 1) \frac{r_i^*}{q^{i+2}} \right],
\]

in a way similar to that in Paper 1. The function \( Q \) is the maximum attainable value of the quantity, \( |\mathbf{q} \times \dot{\mathbf{q}}| \), and the expression for it is derived in the Appendix.

For the proof, we shall assume the opposite and find what condition leads to a contradiction.

Assume that \( \dot{\theta} > 0 \) for all time, \( t > t_0 \). Then, multiplying the expression for \( \ddot{\theta} \) by \( \dot{q} \) and integrating from \( t_0 \) to \( t_1 \), gives

\[
\dot{q}_1^2/2 \leq \frac{GM}{q_1} - \frac{Q^2}{2q_1^2} - GM a M_b \frac{r_1^*}{q_1^3 (q_1 - r_*)} + K,
\]

where

\[
K = \frac{\dot{q}_0^2}{2} - \frac{GM}{q_0} + \frac{Q^2}{2q_0^2} + GM a M_b \frac{r_1^*}{q_0^2 (q_0 - r_*)}.
\]