AUTOMATED, CLOSED FORM INTEGRATION OF FORMULAS IN ELLIPTIC MOTION

WILLIAM H. JEFFERYS
University of Texas at Austin

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Abstract. In some perturbation theories it is possible to avoid expansion of the perturbations in powers of the eccentricity, obtaining results in closed form by using the true or eccentric anomaly instead of the mean anomaly. This paper describes an algorithm (which has been programmed for the 6600 computer using the formula manipulation system TRIGMAN) for automatically performing the integrals which arise in these theories.

Some perturbation theories (e.g. Brouwer, 1959; Hori, 1963; Harrington, 1969; Aksnes, 1969) avoid expansion of the perturbations in powers of the eccentricity by expressing them in terms of the true or eccentric anomaly instead of the mean anomaly. To carry these or other theories to high order, it is necessary to perform the calculations on computers, using a formula manipulation language such as FORMAC (Tobey et. al., 1967), or one of the special purpose systems developed for use in celestial mechanics, such as that of Barton (1967), Deprit and Rom (1968), Hall and Cherniack (1969) or Jefferys (1970).

In the computation of the short-period perturbations, one has to integrate functions of the form

\[ F(r, \dot{r}, u, f) \]  

with respect to the mean anomaly \( l \); where \( r \) is the distance to the primary, \( \dot{r} = dr/dl \), \( u \) is the eccentric anomaly and \( f \) the true anomaly. For convenience we will also assume that the semimajor axis has been normalized to unity.

In hand computations the integrals are computed case-by-case as they arise, using any available method and some well-known tricks. For certain cases, general formulae are available (Kozai, 1962).

On the other hand, when we attempt to automate the calculation, it is necessary to have an algorithm which will invariably work on some well-defined but rather general class of functions. Of course, it is not possible to guarantee that the integrated function will remain in that class of functions, or even that it will itself be integrable in closed form (as will be seen below). Furthermore, it may be necessary to sacrifice elegance and speed in the algorithm in order to obtain reliability and the desired degree of generality.

We will assume that the function to be integrated is a sum of the form

\[ F = \sum P(r, \dot{r}) \frac{\cos (jf + ku + \phi)}{\sin (jf + ku + \phi)}, \]  

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where \( P \) is a polynomial in \( r \) and \( \dot{r}, j \) and \( k \) are integers, and \( \varphi \) is independent of the three anomalies. In general, negative powers of \( \dot{r} \) cannot occur, owing to the resulting singularity that appears at \( l=0, \pi \); but they may occur in Equation (2), provided that they do not occur in Equation (6), below. (For example, the algorithm will successfully handle the function \((\sin u)/\dot{r} = r/e\).)

Also, the fact that we do not normally have both \( j \neq 0 \) and \( k \neq 0 \) in a given term does not affect the algorithm.

The algorithm proceeds as follows:

**Step 1.** Using standard trigonometric formulas and the identities of elliptic motion

\[
\sin f = \dot{r}\eta/e, \\
\cos f = (\eta^2/r - 1)/e, \\
\sin u = rr/e, \\
\cos u = (1 - r)/e,
\]

where \( e \) is the eccentricity and \( \eta = (1 - e^2)^{1/2} \), we may express \( \sin nf, \cos nf, \sin nu, \) and \( \cos nu \) as power series in \( r \) and \( \dot{r} \) for those multiples of \( f \) and \( u \) appearing in the series to be integrated.

**Step 2.** Using the results from Step 1 and the identities

\[
\cos (nf + \Psi) = \cos nf \cos \Psi - \sin nf \sin \Psi, \\
\sin (nf + \Psi) = \sin nf \cos \Psi + \cos nf \sin \Psi,
\]

and similar ones in the eccentric anomaly, we express the function to be integrated as a polynomial in \( r \) and \( \dot{r} \), in which \( u \) and \( f \) do not appear explicitly.

**Step 3.** The identity

\[
(\dot{r})^2 = -1 + 2/r - \eta^2/r^2
\]

is used to eliminate all powers of \( \dot{r} \) higher than the first. Thus (assuming \( F \) does not have any singularities, as mentioned above) we obtain

\[
F = A(r) + B(r) \dot{r},
\]

where \( A \) and \( B \) are polynomials.

**Step 4.** The second term of Equation (6) is separated off and immediately integrated:

\[
\int B(r) \dot{r} \, dl = \int B(r) \, dr.
\]

In this integration it is possible for terms in \( \log(r) \) to appear. Such a term arises, for example, in the third-order Brouwer theory of an artificial satellite, from the term

\[
\frac{\partial F^*}{\partial g} \cdot \frac{\partial S_1}{\partial G},
\]

owing to the fact that \( S_1 \) contains a term in the equation of the center \( f - l \), and

\[
\frac{\partial f}{\partial e} = \left\{ \frac{1}{r} + \frac{1}{\eta^2} \right\} \sin f = \frac{\dot{r}}{\eta e} + \frac{\eta \dot{r}}{r}.
\]