Jacobian criteria for complete intersections.
The graded case*

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Oblatum 14-IV-1993 & 9-IX-1993

Summary. Let $P$ be a positively graded polynomial ring over a field $k$ of characteristic zero, let $I$ be a homogeneous ideal of $P$, and set $R = P/I$. The paper investigates the homological properties of some $R$-modules canonically associated with $R$, among them the module $\Omega_{R|k}$ of Kähler differentials and the conormal module $I/I^2$.

It is shown that a subexponential bound on the Betti numbers of any of these modules implies that $I$ is generated by a $P$-regular sequence. In particular, the finiteness of the projective dimension of the conormal module implies $R$ is a complete intersection. Similarly, the finiteness of the projective dimension of the differential module implies $R$ is a reduced complete intersection. This provides strong converses to some well-known properties of complete intersections, and establishes special cases of conjectures of Vasconcelos.

The proofs of these results make extensive use of differential graded homological algebra. The crucial step is to show that any homomorphism of complexes from the minimal cotangent complex $L_{R|k}$ of André and Quillen into the minimal free resolution of the irrelevant maximal ideal $\mathfrak{m}$ of $R$, which extends the Euler map $\Omega_{R|k} \to \mathfrak{m}$, is a split embedding of graded $R$-modules.

Introduction

The classical jacobian criterion characterizes the regularity of an affine algebra over a field of characteristic zero by the projectivity of its module of (Kähler) differentials. We investigate the relations between the singularity of a graded algebra and the homological properties of its module of differentials, and of various other "cotangent modules" canonically associated with the algebra.

In this paper $k$ denotes a field of characteristic zero, and a graded $k$-algebra $R$ is a graded commutative algebra generated over $k$ by finitely many elements of positive degree. Such an algebra is a quotient of a positively graded polynomial ring $P$ over $k$ by a homogeneous ideal $I$. The ring $R$ is said to be a complete intersection if $I$ can be generated by a regular sequence. This property does not

* Dedicated to Professor Ernst Kunz on his sixtieth birthday
** The first author was partly supported by a grant from NSF. During the preparation of this paper the second author was supported by Purdue University, whose hospitality he wishes to acknowledge
depend on the choice of presentation. The minimal length of a regular sequence required for such a presentation is called the \textit{codimension} of \( R \) and is denoted \( \text{codim} R \). It is known [8] that for a reduced complete intersection \( R \) (even in the affine case) the projective dimension of the module of differentials \( \Omega_{R/k} \) is at most one. We establish a strong converse.

\textbf{Theorem 1} For a graded \( k \)-algebra \( R \) the following conditions are equivalent:

(i) \( R \) is a reduced complete intersection;
(ii) \( \text{projdim}_R \Omega_{R/k} \leq 1 \);
(iii) \( \text{projdim}_R \Omega_{R/k} < \infty \).

A similar characterization of normal complete intersections is given in (2.9); it involves the torsion of the module of differentials. Recall that the \textit{torsion submodule} \( t(M) \) of an \( R \)-module \( M \) consists of all the elements of \( M \) annihilated by non-zero-divisors of \( R \).

In order to describe complete intersections which need not be reduced, we recall that the \( i \)-th \textit{Betti number} of a finite graded \( R \)-module \( M \) is the dimension \( b_i(M) \) of the \( k \)-vector space \( \text{Tor}_i^R(M, k) \), which is equal to the rank of the \( i \)-th module in the minimal homogeneous \( R \)-free resolution \( F(M) \) of \( M \). The rate of growth of the Betti sequence provides a measure of how complex the homological nature of \( M \) is. When comparing rates of growth of two sequences \( \{b_i\}_{i \geq 0} \) and \( \{c_i\}_{i \geq 0} \) we use the notation \( b_i \asymp c_i \) to indicate that \( b_i = c_id_i \) for some sequence \( \{d_i\}_{i \geq 0} \) with \( \lim_{i \to \infty} d_i = 1 \). A numerical measure of the asymptotic rate of growth of the Betti sequence of \( M \) is provided by the radius of convergence \( \rho(M) \) of the Poincaré series
\[
P_M(t) = \sum_{i \geq 0} b_i(M) t^i.
\]

\textbf{Theorem 2} For a graded \( k \)-algebra \( R \) the following conditions are equivalent:

(i) \( R \) is a complete intersection;
(ii) \( b_i(\Omega_{R/k}) \asymp b_i d^{i-1} \) for some \( b \in \mathbb{Q} \) and \( d \in \mathbb{N} \) with \( b > 0 \) and \( \text{codim} R \geq d \geq 0 \);
(iii) \( b_i(\Omega_{R/k}) \leq q(i) \) for some polynomial \( q \in \mathbb{R}[t] \) and all \( i \geq 0 \);
(iv) \( \rho(\Omega_{R/k}) \geq 1 \), that is, the Poincaré series of \( \Omega_{R/k} \) converges in the open unit disk. Furthermore, these conditions are also equivalent to the ones obtained from (ii), (iii), and (iv) by replacing \( \Omega_{R/k} \) with \( \Omega_{R/k}/t(\Omega_{R/k}) \).

If \( e \) is the number of variables of the polynomial ring \( P \), then an exact sequence
\[
1/I^2 \rightarrow R^e \rightarrow \Omega_{R/k} \rightarrow 0,
\]
relates \( \Omega_{R/k} \) and the \textit{conormal module} \( 1/I^2 \) of the embedding \( \text{Spec} R \subseteq \text{Spec} P \). Up to a free direct summand, this graded \( R \)-module is independent of the choice of the embedding; in particular, its positive Betti numbers are invariants of \( R \). They vanish when \( R \) is a complete intersection, as the conormal module is then well known to be free. By Ferrand [8] and Vasconcelos [20] this condition characterizes quite general complete intersections. In the graded case we prove a stronger result, related to a conjecture of Vasconcelos [21].

\textbf{Theorem 3} For a graded \( k \)-algebra \( R \) the following conditions are equivalent:

(i) \( R \) is a complete intersection;
(ii) \( I/I^2 \) is a free \( R \)-module;
(iii) \( \text{projdim}_R I/I^2 < \infty \);
(iv) \( \rho(I/I^2) \geq 1 \), that is, the Poincaré series of \( I/I^2 \) converges in the open unit disk.