Discrete decomposability of the restriction of $A_\theta(\lambda)$ with respect to reductive subgroups and its applications

Toshiyuki Kobayashi*
Department of Mathematical Sciences, University of Tokyo, Meguro, Komaba, 153, Tokyo, Japan
Oblatum 3-IV-1993

Summary. Let $G' \subset G$ be real reductive Lie groups and $\theta$ a $\theta$-stable parabolic subalgebra of $\text{Lie}(G) \otimes \mathbb{C}$. This paper offers a sufficient condition on $(G, G', \theta)$ that the irreducible unitary representation $A_\theta$ of $G$ with non-zero continuous cohomology splits into a discrete sum of irreducible unitary representations of a subgroup $G'$, each of finite multiplicity. As an application to purely analytic problems, new results on discrete series are also obtained for some pseudo-Riemannian (non-symmetric) spherical homogeneous spaces, which fit nicely into this framework. Some explicit examples of a decomposition formula are also found in the cases where $A_\theta$ is not necessarily a highest weight module.

0 Introduction

Our object of study is the restriction of a unitary representation $A_\theta(\lambda)$ of a real reductive linear Lie group $G$ with respect to its reductive subgroup $G'$. Here $A_\theta(\lambda)$ denotes the Hilbert completion of an irreducible unitary $(g, K)$-module $A_\theta(\lambda)$ attached to an integral elliptic orbit $\text{Ad}^*(G) \lambda \subset g^*$ in the sense of Vogan–Zuckerman, which is a vast generalization of Borel–Weil–Bott’s construction of finite dimensional representations of compact Lie groups. It is well-known that the following $(g, K)$-modules are described by means of $A_\theta(\lambda)$ or its coherent family in the weakly fair range (see [V3, Definition 2.5]; §2):

(0.1)(a) representations with non-zero $(g, K)$-cohomology which contributes the de Rham cohomology of locally Riemannian symmetric spaces by Matsushima’s formula (see [BoW, VZ]),
(0.1)(b) discrete series for semisimple symmetric spaces (see [FJ, Chap. VIII, §2; V3, §4]), which include Harish–Chandra’s discrete series for group manifolds,
(0.1)(c) ‘most of’ unitary highest weight modules of classical groups [A2].

Suppose $G' \subset G$ are Lie groups, $X$ is a $G$-space and $X'$ is a $G'$-space. Then a representation theoretic counterpart of an equivariant morphism $f: X' \to X$ is...
the pullback of function spaces \( f^*: \Gamma(X) \to \Gamma(X') \), where the restriction of representations of \( G \) with respect to \( G' \) naturally arises. If \( A_\eta(\lambda) \) is realized in a function space \( \Gamma(X) \) as in (0.1)(a) and (b), it is natural to ask the restriction formula (branching rule) of \( A_\eta(\lambda)_{|G'} \) into irreducible representations of \( G' \). So far, the following special cases of the restriction of \( A_\eta(\lambda) \) with respect to reductive subgroups have been achieved (see also Examples 4.5 and 4.6):

(0.2)(a) \( G \) is compact. A classical (but still active) study of branching rules of finite dimensional representations of compact Lie groups ('breaking-symmetry' in physics) is to find explicit restriction formulas with respect to various subgroups.

(0.2)(b) \( G' \) is a maximal compact subgroup of \( G \). An explicit decomposition formula is known as a generalized Blattner formula (see [HS; V1 Theorem 6.3.12]).

(0.2)(c) \( A_\eta(\lambda) \) is of a highest (or lowest) weight. An explicit decomposition formula is found (e.g. [M, J, JV]) in the cases where \( A_\eta(\lambda) \) is holomorphic discrete series with some assumption on \( G' \) (see the condition (4.1)(a')).

However, one can observe that the restriction of \( A_\eta(\lambda) \) with respect to \( G' \) may have a wild behavior in general, even if \( G' \) is a maximal reductive subgroup in \( G \), involving the following cases:

(0.3)(a) The restriction is decomposed into only the continuous spectrum with infinite multiplicity (e.g. a tensor product of principal series of simple complex groups other than \( \text{SL}(2, \mathbb{C}) \); see [GG, Wi]).

(0.3)(b) The restriction is decomposed into the continuous spectrum with finite multiplicity and at most finite many discrete spectrum (possibly no discrete spectrum) (e.g. the tensor product of a holomorphic discrete series and an anti-holomorphic discrete series [R]).

(0.3)(c) The restriction is decomposed into countably many discrete spectrum with finite multiplicity (see §3, §4, §6.1).

(0.3)(d) The restriction is still irreducible (e.g. Theorem 6.4).

For a fruitful study of the restriction of \( A_\eta(\lambda) \) with respect to a reductive subgroup in a general setting, we first want to find a good framework, where we can expect to obtain explicit and informative branching rules which are not only interesting from view points of representation theory but also applicable to harmonic analysis as in the situations of (0.1)(a) and (b). For this purpose we focus our attention to the case where the restriction is an 'admissible' representation. Here, we say a unitary representation \((\pi, V)\) of \( G \) is \( G \)-admissible if \((\pi, V)\) is decomposed into a discrete Hilbert direct sum with finite multiplicities of irreducible representations of \( G \). Previous examples (0.3)(c) and (d) are the case. Successful theories (0.2)(a) \sim (c) are also the case. One of the advantages of admissibility is to allow one to study algebraically the objects in which such representation \((\pi, V)\) occurs. We also illustrate this in some other familiar results which have laid important foundations on the study of locally symmetric spaces (e.g. [BoW, VZ]), algebraic study of Harish-Chandra modules (e.g. [V1]).

(0.4)(a) (induction) Let \( \Gamma \) be a cocompact discrete subgroup of \( G \). Then the \( L^2 \)-induced module \( L^2-\text{Ind}^G_\Gamma(1) = L^2(G/\Gamma) \) is \( G \)-admissible (Gelfand and Piatecki–Sapiro, [GGP, Chap. 1, §2]).

(0.4)(b) (restriction) Let \( K \) be a maximal compact subgroup of a reductive linear Lie group \( G \) and \((\pi, V)\) an irreducible unitary representation of \( G \). Then the restriction \((\pi_{|K}, V)\) is \( K \)-admissible (Harish–Chandra).