LINEARIZATION IN SPECIAL CASES OF PERTURBED KEPLERIAN MOTIONS

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Abstract. This paper treats linearization of problems of motion in three-dimensions in not-central force fields, when the problems are reducible to the one-dimensional case by using integrals of the motion. Linearizing transformations of the independent variable are found to solve such problems if the motion is bounded, and explicit forms of the regularizing functions, corresponding to more common potentials, are given. An application is presented to the integration of a radial intermediary orbit that arises in the analytical study of the theory of artificial satellites.

1. Introduction

Regularization and its special case, linearization, are becoming increasingly powerful techniques in dynamics. When applied to problems of Celestial Mechanics, they permit the removal of the Lyapunov instability of the two-body problems, and the obtention of the suitable properties of numerical or analytical stability of linear oscillators.

For the last years, a number of researches on these methods have been carried out. We mention, as general references, the work of Stiefel and Scheifele (1971) or Szebehely (1976), and, more closely related with this paper, the results of Belen’kii (1981), Cid et al. (1983) and Caballero and Ferrer (1983), which pay special attention to central force fields.

On the other hand, recent work developed in a rather remote framework (Deprit, 1981), suggests the convenience of introducing, in the theory of the artificial satellite, radial or zonal intermediaries beyond the unperturbed one. All those ideas lead us to apply linearization methods to study problems in which the force fields are not central, but have a kind of angular momentum integrals, associated with cyclical coordinates.

In this paper, we deal with the question of linearizing a wide class of such problems, by means of transformations of the independent variable. Assuming that the qualitative behavior of orbits must be conserved, we establish a criterion for the existence of linearizing functions, and their computation for elliptic-type motions. The method is particularized, and explicit forms of regularizing functions given, to the case when the potential may be cast as a finite series of positive and negative integer powers of the radius. Finally, an application concentrates on the problem of the artificial satellite, carrying out the complete integration of the Cid’s radial intermediary.
2. Basic Relations

Let us consider the motion of a particle of reduced mass $\mu$, in a conservative field with potential $V_1$. In Whittaker (1904) canonical variables $(r, \theta, v, R, \Theta, N)$, where $r$ denotes the radial length, $\theta$ the polar angle in the osculating orbital plane measured from the rising node, $v$ the longitude of the rising node, and $R, \Theta, N$ their respective conjugated momenta, the Hamiltonian may be written as

$$H = \frac{1}{2} \left( R^2 + \frac{\Theta^2}{r^2} \right) - \frac{\mu}{r} + V_1. \quad (1)$$

We assume throughout the paper that potential $V_1$ depends on the single coordinate $r$ and possibly on both momenta $\Theta$ and $N$. Such a Hamiltonian belongs to the class of the so-called radial intermediaries (Deprit, 1981) which may often be found in the analytical integration by using perturbations methods of orbital problems.

Besides $\Theta, N$, the problem has the energy as first integral. The latter can be written as

$$\frac{1}{2} R^2 = \frac{1}{2} \left( \frac{dr}{dr} \right)^2 = h - V. \quad (2)$$

where $h$ is the total energy and $V$ stands for the effective potential

$$V = V_1 - \frac{\mu}{r} + \frac{1}{2} \frac{\Theta^2}{r^2}. \quad (3)$$

Let us introduce a new independent variable $s$ by means of the differential relation

$$dt = g(r) \, ds, \quad (4)$$

$g(r)$ being strictly positive and continuously differentiable along trajectories. In terms of the fictitious time $s$, the equations of motion for the angular variables are

$$\frac{d\theta}{ds} = g \frac{\partial H}{\partial \Theta}, \quad \frac{dv}{ds} = g \frac{\partial H}{\partial N} \quad (5)$$

while the equation for the radial variable, $r$, is replaced by either

$$\frac{1}{2} \left( \frac{dr}{ds} \right)^2 = g^2(h - V) \quad (6)$$

or the second order differential equation

$$\frac{d^2r}{ds^2} = \frac{\partial}{\partial r} \left[ g^2(r)(h - V(r)) \right]. \quad (7)$$

Therefore, transformation (4) leads to a linear equation for $r$ if and only if there exists constants, $c_1$, $c_2$ and $c_3$, such that the regularizing function $g$ satisfies the identity

$$g^2(r)(h - V(r)) = \frac{1}{2} c_1 r^2 + c_2 r + c_3 \quad (8)$$