CHAOS IN A QUARTIC DYNAMICAL MODEL

N. CARANICOLAS and CH. VOZIKIS

Department of Astronomy, University of Thessaloniki 54006 Thessaloniki, Greece

(Received 10 July, 1986; accepted 16 January, 1987)

Abstract. We study the orbital characteristics of a time independent, two dimensional quartic dynamical model with two exact periodic orbits that displays always closed zero velocity curves. It is shown that the stability of the periodic orbits depends on the value of the coupling parameter $\alpha$. Computer calculations suggest that the degree of stochasticity is small for the values of $\alpha$ in the range $1 < \alpha < 3$ while it grows rapidly when $\alpha > 3$. We also compute the Lyapunov characteristic exponents for different values of the coupling parameter.

1. Introduction

A Hamiltonian system of $n$ degrees of freedom is integrable if $n$ global isolating integrals of motion exist. The integrals must be in involution i.e. with vanishing Poisson brackets. These integrals restrict the phase space trajectories of the system to $n$-dimensional tori.

But integrable Hamiltonian systems are extremely rare. In general two or higher-dimensional Hamiltonian systems are non-integrable and display a complicated phase space structure.

On the other hand, if a Hamiltonian system is near-integrable, in the sense that it can be treated as a perturbed integrable system, then according to the KAM theorem (Kolmogorov 1954, Arnold 1963, Moser 1962) most of the tori that compose the phase space are merely distorted by the perturbation. The corresponding phase space trajectories are regular, i.e. are associated with first integrals of motion, while the remaining trajectories exhibit stochastic or chaotic behaviour.

In this article we investigate the properties of the homogeneous quartic dynamical model

$$V = x^4 + y^4 + 2\alpha x^2 y^2,$$  \hspace{1cm} (1)

where $\alpha > 1$. In particular we search whether this system has small or large stochastic regions and how the degree of stochasticity is connected with the value of the coupling parameter $\alpha$.

The paper is organized as follows. In Section 2 we study the form of the zero velocity curves. In Section 3 we find the periodic orbits and study their stability. Section 4 describes the surfaces of section and the transition from regular to stochastic motion as the coupling parameter increases. Furthermore in this Section we compute the Lyapunov characteristic exponents in the regular as well as in the...
stochastic regions for different values of \( \alpha \). Finally in Section 5 we present a discussion and the conclusions of this work.

2. Zero Velocity Curves

The Hamiltonian to the potential (1) is

\[
H = \frac{1}{2}(x^2 + y^2) + x^4 + y^4 + 2\alpha x^2 y^2 = h, \tag{2}
\]

where the dot indicates derivative with respect to the time and \( h \) is the numerical value of \( H \).

The zero velocity curves (ZVC) are given by the equation

\[
x^4 + y^4 + 2\alpha x^2 y^2 - h = 0. \tag{3}
\]

Since (3) has no saddle points for positive values of \( \alpha \) (Caranicolas and Varvoglis, 1984) the ZVC are always closed curves. In the integrable case \( \alpha = 1 \) they are circles around the origin. We now proceed to study the form of the ZVC when \( \alpha > 1 \) as \( \alpha \) increases.

Equation (3) defines \( y \) as an implicit function of \( x \). Then the curvature of the curve (3) is

\[
\sigma = \frac{-y''}{[1 + y'^2]^{3/2}}, \tag{4}
\]

where the prime indicates derivative with respect to \( x \). Since our model is symmetric with respect to both axes it suffices to study the properties of the ZVC when \( x > 0 \), \( y > 0 \). Therefore (4) gives

\[
\sigma = \frac{[(3 - \alpha^2)x^2 y^2 + \alpha(x^4 + y^4)]V}{[x^6 + y^6 + \alpha(\alpha + 2)(x^2 + y^2)x^2 y^2]^{3/2}}, \tag{5}
\]

An elementary analysis in (5) shows that when \( 1 < \alpha < 3 \), \( \sigma \) is positive at any point of the curve except the point \( x = y \) for \( \alpha = 3 \) where \( \sigma = 0 \). On the other hand when \( \alpha > 3 \) the curvature is negative in the range between the points

\[
x = \frac{h^{1/4}}{(1 + 2\alpha p^2 + p^4)^{1/4}}, \quad y = \frac{h^{1/4}p}{(1 + 2\alpha p^2 + p^4)^{1/4}},
\]

and

\[
x = \frac{h^{1/4}p}{(1 + 2\alpha p^2 + p^4)^{1/4}}, \quad y = \frac{h^{1/4}}{(1 + 2\alpha p^2 + p^4)^{1/4}},
\]

where

\[
p^2 = \frac{(\alpha^2 - 3) - (\alpha^4 - 10\alpha^2 + 9)^{1/2}}{2\alpha}, \tag{7}
\]

while it is positive elsewhere. It is worth to notice that as \( \alpha \) increases, the range