THE TWO-BODY PROBLEM IN THE (TRUNCATED) PPN – THEORY *

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Abstract. The solution of the two-body problem in the (truncated) PPN theory is presented. It is given in two different analytical forms (the Wagoner-Will and Brumberg representation) and by the method of osculating elements.

1. Introduction
Analyzing gravitational experiments in the solar system is usually done in the so-called PPN - framework (e.g. Will 1981), where a number of PPN - parameters designate the corresponding post - Newtonian limit of a certain metric theory of gravity. Now, the discovery of the binary pulsar PSR1913+16 (e.g. Taylor & Weisberg 1982) and subsequent extremely precise tracking of its orbital motion by analyzing pulse arrival times lead to the necessity to solve for the full two - body problem at least at the post - Newtonian level. For the Einstein post- Newtonian theory one solution to the two - body problem has been presented by Wagoner & Will (1976), Epstein (1977) and Haugan (1985); a solution with osculating elements for this case was presented by Damour & Deruelle (1985). In a series of papers Barker & O’Connell (1975, 1976, 1981) and Barker et al. (1982, 1986) dealt with the full post - Newtonian two - body problem even including spin and quadrupole moment effects. However, their main interest was lying in the precession and nutations of the spins and the secular motions of the classical angular momentum vector, the Runge - Lenz vector and the mean anomaly rather than solving for the detailed motions of the bodies.

This paper presents solutions to the full two- body problem in the (truncated) PPN - framework with parameters $\beta$ and $\gamma$. Solutions are given in two different analytical forms (the Wagoner-Will and Brumberg representation) and by the method of osculating elements.

The Lagrangian for the two - body problem in the PPN - formalism truncated to the Eddington - Robertson parameters $\beta$ and $\gamma$ in standard post - Newtonian coordinates $(t, \mathbf{x})$ reads (e.g. Will 1981):

\[ \mathcal{L} = -(m_1 + m_2)c^2 + \mathcal{L}_N + \mathcal{L}_{PN}/c^2 \]

\[ \mathcal{L}_N = \frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2 + \frac{Gm_1m_2}{r} \]

\[ \mathcal{L}_{PN} = \frac{1}{8} m_1 v_1^4 + \frac{1}{8} m_2 v_2^4 + \frac{Gm_1m_2}{2r} \left[(2\gamma + 1)(v_1^2 + v_2^2) - (4\gamma + 3)v_1 \cdot v_2 - (v_1 \cdot \hat{n})(v_2 \cdot \hat{n}) - (2\beta - 1)\frac{G(m_1 + m_2)}{r}\right] \]

with

\[ \hat{n} = \frac{\mathbf{x}_1 - \mathbf{x}_2}{r}; \quad r = |\mathbf{x}_1 - \mathbf{x}_2| \]

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One finds that the total momentum $P$ of the system can be obtained in the usual way from $\partial L/\partial v_1 + \partial L/\partial v_2$ and is given by:

$$P = m_1 v_1 + m_2 v_2 + \frac{1}{2} m_1 v_1^2 / c^2 + \frac{1}{2} m_2 v_2^2 / c^2 + \frac{G m_1 m_2}{2c^2 r} [2(2\gamma + 1)(v_1 + v_2) - (4\gamma + 3)(v_1 + v_2) - \hat{n} \cdot (v_1 + v_2)]$$

(2)

The center of mass $X$

$$X = (m_1^* x_1 + m_2^* x_2) / (m_1^* + m_2^*)$$

(3)

with

$$m_a^* = m_a + \frac{1}{2} m_a v_a^2 / c^2 - \frac{1}{2} G m_1 m_2 / r$$

(4)

then is not accelerated according to the equations of motion and the center of mass velocity is proportional to $P$. We can then go to a post - Newtonian center of mass frame where $P = X = 0$

and

$$x_1 = \frac{m_2}{m} \left[ \frac{\mu \delta m}{2m^2} (v^2 - \frac{G m}{r}) \right] x$$

(5a)

$$x_2 = \left[ -\frac{m_1}{m} + \frac{\mu \delta m}{2m^2} (v^2 - \frac{G m}{r}) \right] x$$

(5b)

with

$$x = x_1 - x_2 ; \quad v = v_1 - v_2 ; \quad m = m_1 + m_2$$

$$\delta m = m_1 - m_2 ; \quad \mu = m_1 m_2 / m$$

For the relative motion one finds (e.g. Barker et al. 1986):

$$\frac{dv}{dt} = -\frac{G m \hat{n}}{r^2} + G m \hat{n} \left\{ \frac{G m}{r} (2(\beta + \gamma) + 2\nu) - v^2 (\gamma + 3\nu) + \frac{3}{2} \nu (\hat{n} \cdot v)^2 \right\}$$

$$+ \frac{G m}{c^2 r^2} v (\hat{n} \cdot v) (2\gamma + 2 - 2\nu)$$

(6)

and the corresponding Lagrangian takes the form:

$$L = \frac{1}{2} v^2 + \frac{G m}{r} + \frac{1}{8} (1 - 3\nu) v^4 + \frac{G m}{c^2 r^2} \left[ (2\gamma + 1 + \nu) v^2 + \nu (\hat{n} \cdot v)^2 - (2\beta - 1) \frac{G m}{r} \right]$$

(7)

This Lagrangian is particularly useful in deriving first integrals of motion. For the (specific) post - Newtonian energy $E$ and angular momentum $J$ one finds:

$$E = v \frac{\partial L}{\partial v} - L = \frac{1}{2} v^2 - \frac{m}{r} + \frac{3}{8} (1 - 3\nu) v^4 + \frac{m}{2r} \left[ (2\gamma + 1 + \nu) v^2 + \nu (\hat{n} \cdot v)^2 + (2\beta - 1) \frac{m}{r} \right]$$

(8)

and

$$J = |x \wedge \frac{\partial L}{\partial v}| = |x \wedge v| \left[ 1 + \frac{1}{2} (1 - 3\nu) v^2 + (2\gamma + 1 + \nu) \frac{m}{r} \right]$$

(9)