RESEARCHES ON THE RESTRICTED THREE-BODY PROBLEM

II. Periodic Solutions and Arcs for \( \mu = 0^* \)

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Abstract. This work considers periodic solutions and arc-solutions (solutions with consecutive collisions) of the plane circular restricted problem of three bodies for \( \mu = 0 \). To study the mutual arrangement of these solutions in phase space, we introduce a global section \( \Gamma \) of the phase space. Each solution is represented in \( \Gamma \) by some points (at least one), and families of solutions are represented by curves (the characteristics). We give a concrete description of the arrangement of the characteristics in the three-dimensional space \( \Gamma \). The solutions studied here play a fundamental role in generating periodic solutions of the restricted problem for small \( \mu \neq 0 \).

0. Section of Phase Space

0.A. LOCAL SECTION

We consider a Hamiltonian system with two degrees of freedom

\[
\begin{align*}
\dot{x}_i &= \frac{\partial H}{\partial y_i}, \\
\dot{y}_i &= -\frac{\partial H}{\partial x_i},
\end{align*}
\]

where the Hamiltonian \( H(x_1, x_2, y_1, y_2) \) is analytic in a domain \( G \) of the four-dimensional space of the variables \( x_1, x_2, y_1, y_2 \). Let \( M \) be a periodic solution of this system, lying in the domain \( G \) together with its neighborhood \( U \), and let \( \Gamma \) be a hypersurface passing through some point \( P \) of the solution \( M \).

If the solution \( M \) intersects the hypersurface \( \Gamma \) transversally at the point \( P \), i.e. if the vector formed by the left-hand members of the system (0.1) at the point \( P \) does not lie in the hyperplane tangent to \( \Gamma \), then \( \Gamma \) is intersected by all periodic solutions of the family \( F \) of which the solution \( M \) is a member (see Brjuno, 1972) which are close to \( M \). The intersection \( \Gamma \cap F \) is a one-dimensional curve; it lies in \( \Gamma \cap U \) like a curve \( \Sigma \) in a three-dimensional neighborhood \( V \) (Brjuno, 1972). Such a curve, obtained as the intersection of a family \( F \) of periodic solutions and some secant hypersurface, will be called a characteristic of the family \( F \). Thus, in the transversal case, the characteristic of family \( F \) in some neighborhood \( \Gamma \cap U \) of the point \( P \) is a simple curve, passing through the point \( P \). If \( M \) is a resonant solution, then in the neighborhood \( U \) lie families \( F_i \) or \( F_* \) of multiple periodic solutions, and their characteristics lie in \( \Gamma \cap U \) like a set of curves \( \Sigma_i \) in a neighborhood \( V \) (Brjuno, 1972).

If the Hamiltonian \( H \) and the solution \( M \) are symmetric, and the point \( P \) lies in the plane of symmetry \( \pi \), then some of the characteristics will lie in the plane of symmetry \( \pi \) (in Section 5 (Brjuno, 1972), these characteristics are called curves \( \sigma \)).

We consider now the case where the periodic solution \( M \) is tangent to the hypersurface \( \Gamma \) at the point \( P \). Such a point \( P \) will be called exceptional for the section \( \Gamma \). Exceptional points form in \( \Gamma \) a two-dimensional sub-variety \( \Gamma_* \), given by the system

\[
\gamma(x_1, x_2, y_1, y_2) = 0, \\
\sum_{i=1}^{2} \left( \frac{\partial \gamma}{\partial x_i} \frac{\partial H}{\partial y_i} - \frac{\partial \gamma}{\partial y_i} \frac{\partial H}{\partial x_i} \right) = 0, \tag{0.2}
\]

where \( \gamma = 0 \) is the equation of the hypersurface \( \Gamma \). Here we shall consider only the non-resonant solution \( M \) of the general position. Then a unique family \( F \) of periodic solutions passes through \( M \). If \( P \) does not lie in the plane of symmetry \( \pi \), the characteristic \( \chi F \) of the family \( F \) in \( \Gamma \cap U \) has the shape of a parabola with its top in the point \( P \); therefore a solution of \( F \) is represented on \( \chi F \) either by two points, lying on opposite sides of the top, or by no point at all. If the point \( P \) is in the plane of symmetry \( \pi \) of the Hamiltonian \( H \), and if the solution \( M \) is symmetric, then the family \( F \) has in \( \Gamma \cap U \) two characteristics: one of them (which we shall call \( \chi_0 F \)) lies in the plane of symmetry \( \pi \), and each solution in \( \Gamma \cap U \) has one point on \( \chi_0 F \); the other characteristic \( \chi_1 F \) of the family \( F \) has the form of a parabola with its top in the point \( P \) (Figure 0). Therefore the solutions lying on \( F \) on one side of \( M \) are represented on the characteristic \( \chi_1 F \) by two points \( Q_1 \) and \( Q_2 \) (Figure 0); the solutions lying on \( F \) on the other side of \( M \) are not represented at all on \( \chi_1 F \).

0.B. GLOBAL SECTION

Suppose that for the system (0.1) one has constructed a secant hypersurface \( \Gamma \) in phase space, such that every periodic solution intersects \( \Gamma \), and the plane of symmetry \( \pi \) belongs to \( \Gamma \). Then each natural family \( F \) of periodic solutions is represented in \( \Gamma \) by its characteristics, which in general there can be more than one. Therefore the mutual arrangement in \( \Gamma \) of the characteristics of the various families corresponds to the mutual arrangement of these same families in phase space. In this way, the section \( \Gamma \) allows a reduction by one unit of the dimension of all objects under study, preserving their mutual arrangement and making them more concrete. Characteristics of the families in the plane of symmetry have the greatest concreteness. A global section \( \Gamma \) has in general a two-dimensional subset \( \Gamma_* \) of exceptional points, which are the solutions of the system (0.2).

For the system (0.1) in the domain \( G \), a solution can either be continued on both sides (for any finite value of the time \( t \), the solution remains in \( G \)), or it can be continued only on one side, or it cannot be continued on both sides (the variation of the time \( t \) on the solution does not exceed some finite value). Solutions of the last kind will be called arcs.